National Academy of Sciences of Ukraine
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Insight into Modules over Dedekind Domains

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In memoriam Dmitri I. Zatsev

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Insight into Modules over Dedekind Domains

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In this book the authors collected and systematized some fundamental results on modules over Dedekind domains. In particular, the results about the free and projective modules, injective and divisible modules, finitely generated and artinian modules, direct sums of cyclic modules, and basis modules were placed here.

The book may be useful for researchers and graduate students who will be willing to study this subject. It can also serve as a suitable textbook for a lecture course in modules over Dedekind domains.

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У книзі автори зібрали та систематизували деякі фундаментальні результати про модулі над дедекіндовими областями. Зокрема, результати про вільні та проективні модулі, ін'єктивні та ділимі модулі, скінченно породжені та артинові модулі, прямі суми циклічних модулів та базисні підмодулі, представлені тут.

Книга може бути корисною як для дослідників, так і для аспірантів та студентів старших курсів, які забажають вивчити ці суб'єкти. Її також можна використати як основу для курсу лекцій про модулі над дедекіндовими областями.

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Preface

The concept of a module over a ring is a natural generalization of the concept of a vector space. This generalization is very broad, and therefore in the general case, it would be too optimistic to hope for development of the theory of modules over arbitrary rings. Figuratively speaking, if a ring R is quite distinct (by its properties) from being a field, then structure of modules over R is very sophisticated. On the other hand, every abelian group can be considered as a module over the ring $\mathbb{Z}$ of integers. Thus, the theory of modules can be viewed as a generalization of the theory of abelian groups. The latter is one of the most developed branches in Algebra (see, for example, the books [**FL1970, FL1973, KI1969, KU1967**]). Consequently, one of the natural ways of the development of Module Theory was determined by the search for analogies of fundamental results on abelian groups. More precisely, each of these fundamental results generates a quest for a maximally wide extension R of the ring $\mathbb{Z}$, such that this specific results (perhaps, appropriately modified) is valid for modules over R. Many key concepts of modern Module Theory and some new interesting types of rings have been introduced on this path. Notice here the following remarkable pattern: Dedekind domains are the largest class of rings such that almost all natural basic results on abelian groups have corresponding analogies for modules over them.

The idea of Dedekind domain evolved in process of development of Algebraic Number Theory. In his famous Supplement XI to the Dirichlet book [**DE1871**], R. Dedekind proved that every ideal of the ring of integers of the fields of algebraic numbers is decomposable into prime factors and such factorization is unique. He introduced the fractional ideal and proved that a non-zero fractional ideal formes an abelian group. These properties determine a Dedekind domain. Note, that Dedekind domains are quite wide extensions of the ring $\mathbb{Z}$. We will justify this fact later by many examples. It is worthwhile to note in addition that the aforementioned above approach has led to new characterizations of Dedekind domains. Thus, for instance, the following statements are equivalent for an integral domain R: (1) R is a Dedekind domain; (2) every divisible R – module is injective; (3) R is a hereditary ring; (4) if A is an R – module such that $Ann_R(A) \neq < 0 >$, then A is a direct sum of cyclic submodules.

Dedekind domains also play an important role in other branches of Modules Theory. In particular, their influence is especially significant in the theory of modules over group rings. The theory of (generalized) soluble groups is one of the origins of the theory of modules over group rings. Modules appear here in the following understandable way. If G is a group, A is a normal abelian subgroup of A, $H = G/A$, then we can think of A as a

module over the integral group ring $\mathbb{Z}H$. Here we define the action $\mathbb{Z}H$ on A by the rule

$$a(n_1 y_1 + \ldots + n_k y_k) = (a^{x_1})^{n_1} \ldots (a^{x_k})^{n_k}, \text{ where } a \in A,\ y_j = x_j A,\ 1 \le j \le k.$$

This approach became standard in Finite (Soluble) Group Theory, and it successfully generated many important results. This progress unsurprisingly led to effective applications of Ring Theory and Module Theory to Infinite (Generalized Soluble) Group Theory; in particular, to those groups that are in any sense considerably close to finite groups, namely, to the groups with finiteness conditions. The famous Hall papers [**HP1954, HP1959, HP1961**] shaped a very resourceful stage in the process of implementation of modules methods in groups with finiteness conditions. The following key result due to P. Hall was a foremost one at this first stage: if R is a noetherian ring and G is a polycyclic -- by – finite group, then a group ring RG is also noetherian. This theorem formed a basis for the well – developed now the theory of group rings over polycyclic – by – finite groups (see the books [**PD1971, PD1977**] and surveys [**PD1975, PD1984, ROJ1985, FD1987**]). This opened, in turn, some bright perspectives for development of the theory of modules over group rings of polycyclic – by – finite groups. Furthermore, some other problems of Group Theory and Modules Theory made necessary the study of modules over group rings for other types of groups. For these group rings, such significant finiteness restrictions as, for example, being noetherian are not valid. Moreover, in many cases, the structure of these group rings is quite complicated. In such cases, the scalar ring R has key influence on the structure of a module A over a group ring RG. Therefore, one can expect a real progress in exploration of modules over the group ring RG only if modules over the correspondent scalar ring R have been studied well enough. It happens to be in the case when R is a Dedekind domain.

Working with modules over group rings, the authors were surprised to find that the necessary information on modules over Dedekind domains is not gathered in one place but rather distributed over a quite variety of research articles. Moreover, in some works this information could be extracted as consequences from general results. Therefore, we decided to collect and systematize some fundamental results on modules over Dedekind domains in this book. This set of results is far from being fully completed. We did not pursue such an ambitious goal since it would require a book of a much bigger volume. The selection of the results was determined by our personal tastes and experience. However, the most important results (about free and projective modules, injective and divisible modules, finitely generated and artinian modules, direct sums of cyclic modules, and basis modules) were placed here. We hope the book may be useful for researchers and graduate students who will be willing to study this subject. This book can also serve as a suitable textbook for a lecture course in modules over Dedekind domains.

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Chapter 1

Dedekind Domains

The idea of the Dedekind domain evolved in the process of the development of Algebraic Number Theory. In his famous Supplement XI to the Dirichlet book [DE1871], R. Dedekind proved that every ideal of the ring of integers of the fields of algebraic numbers is decomposed into prime factors and that factorization is unique. He created the fractional ideal and proved that non-zero fractional ideal formed an abelian group. Nowadays there are many definitions of Dedekind domains. We begin with the following.

We will consider an integral domain R and denote by K the field of fractions of R. A non – zero R – submodule A of K is called a *fractional ideal of* R if $aA \leq K$ for a certain non – zero element $a \in R$. This definition shows that all elements of A have a "common denominator," $a \in R$. If the ring R includes a fractional ideal A, then A is an ideal in the common sense. The converse holds for non-zero ideals. If A and B are fractional ideals, then their *product* is defined as the set of all sums

$$a_1b_1 + \ldots + a_mb_m \quad \text{where } a_i \in A,\ b_j \in B,\ 1 \le j \le m.$$

This set is also a fractional ideal. Clearly, AB is a non – zero R – submodule of K. If x, y are non – zero elements of R such that $xA \le R$ and $yB \le R$. then for every element c of the form $c = a_1b_1 + \ldots a_mb_m$ we have

$$(xy)c = xy(a_1b_1 + \ldots + a_mb_m) = (xa_1)(yb_1) + \ldots + (xa_m)(yb_m) \in R.$$

Obviously $A(BC) = (AB)C$ for any fractional ideals A, B and C of R. This means that the set $\mathcal{FI}(R)$ of all fractional ideals of R is a commutative semigroup and R is its identity element.

A fractional ideal A is said to be a *principal fractional ideal*, if $A = xR$ for some element $0 \neq x \in K$.

If A is a fractional ideal of R, then put

$$A' = \{ x \in K \mid xA \le R \}.$$

1.1. Lemma. *Let R be an integral domain, A a fractional ideal of R. Then A' is a fractional ideal of R. Furthermore, $AA' \le R$.*

Proof. Let $x, y \in A'$, $a \in A$, then $(x - y)a = xa - ya \in$ A, that is, $x - y \in A'$. If $z \in R$, then $(zx)a = z(xa) \in R$. It follows that A' is an R–submodule of K. Choose a non – zero element $u \in R$ with the property $uA \in R$. If $a \in A$, then $ua \in R \le A$, and for arbitrary $x \in A'$ we have $xua = (xa)u \in R$. Thus A' is a fractional ideal of R. The inclusion $AA' \le R$ immediately follows from the definition.

Let A be a fractional ideal of R. We say that A *is invertible*, if $AA' = R$. In this case, we will write A^{-1} instead of A'.

1.2. Proposition. *Let R be an integral domain. Then*

(1) Every principal fractional ideal of R is invertible.

(2) The family of all invertible fractional ideal of R is a group under multiplications.

Proof. If $A = aR$, then evidently $A' = a^{-1}R$ and $AA' = R$. Observe that the equality $AB = R$ implies $B = A^{-1}$. In fact, we have $B \le A'$. Thus $R = AB \le AA' \le R$ and hence $AA' = R$. Moreover,

$$A' = A'R = A'AB = RB = B.$$

The invertibility of the product of invertible fractional ideals and the equality $A = (A^{-1})^{-1}$ are obvious.

Let R be a commutative ring. We define the *prime spectrum Spec(R)* or just the *spectrum* of R to be the set of all prime ideals of R.

An integral domain D is called a *Dedekind domain*, if every its fractional ideal is invertible. In other words, a semigroup $\mathcal{FI}(R)$ is a group.

The assertion (1) of Proposition 1.2 implies that every principal ideal domain is a Dedekind domain and, in particular, the ring $\mathbb{Z}$ of all integers is Dedekind.

Now we will list the most important properties of Dedekind domains. In particular, we will show that every Dedekind domain is a noetherian ring.

Recall some necessary concepts.

An ordered set $\mathcal{P}(\mathcal{P} \leq)$ is said to *satisfy the maximal condition* if every non – empty subset of $\mathcal{P}$ has a maximal element.

It is easy to see that $\mathcal{P}$ satisfies the maximal condition if and only if $\mathcal{P}$ satisfies *the ascending chain condition*, that is, given an ascending chain

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$$

of elements of $\mathcal{P}$ there is some $k \in \mathbb{N}$ such that $a_k = a_{k+n}$ for any $n \in \mathbb{N}$.

For a ring R and an R – module A, we consider the lattice $\mathcal{L}_R(A)$ consisting of all R –submodules of A ordered by inclusion. We say that A is *a noetherian R – module* or it satisfies *the maximal condition on its submodules* if $\mathcal{L}_R(A)$ satisfies the maximal condition.

As we mentioned above, this is equivalent to the statement that members of $\mathcal{L}_R(A)$ satisfying the ascending chain condition.

A ring R is called *right noetherian* if the right R – module R is noetherian. Similarly, we define *left noetherian* ring. We observe that for the commutative rings the concepts of right noetherian and left noetherian rings are coincide. Since in this book we will consider modules over the commutative rings, we will speak simply about noetherian rings.

Every field and the ring $\mathbb{Z}$ of all integers are examples of noetherian rings. By Hilbert's Basis Theorem the ring of polynomials $R[X]$ over a commutative noetherian ring R is likewise noetherian. As a consequence, we obtain that any commutative ring R which is finitely generated over its noetherian subring S with the same identity is noetherian. Moreover, the ring of formal power series $R[[X]]$ over a commutative noetherian ring is likewise noetherian.

Let R be a ring and let A be an R – module. Let B be a subset of A. We define *the R – annihilator of B* as the left ideal of R

$$Ann_R(B) = \{ x \in R \mid ax = 0 \text{ for all } a \in B \}.$$

In particular, the annihilator of A is the two – sided ideal and $Ann_R(A) = \bigcap_{a \in A} Ann_R(a)$.

1.3. Proposition. *Let R be a ring and A an R – module.*

(1) If A is noetherian and B a submodule of A, then B and A/B are noetherian.
(2) If B is a submodule of A such that B and A/B are noetherian, then A is noetherian.
(3) If A has a finite series of submodules

$$< 0 > = A_0 \leq A_1 \leq \ldots \leq A_n = A$$

every factor of which is noetherian, then A is likewise noetherian.

(4) If $A = A_1 + \ldots + A_n$ and $A_1, \ldots, A_n$ are noetherian, then A is noetherian.
(5) A is noetherian if and only if every submodule of A is finitely generated.
(6) R is noetherian ring if and only if every right ideal of R is finitely generated.
(7) If R is a noetherian ring, then any finitely generated R-module is noetherian.

Proof. (1) is obvious. (3) is an immediate consequence of (2).

(2) Let

$$A_1 \leq A_2 \leq \ldots \leq A_n \leq \ldots$$

be an ascending chain of submodules of A. Since B and A/B are noetherian, there is some $m \in \mathbb{N}$ such that $A_m \cap B = A_{m+n} \cap B$ and $A_m + B = A_{m+n} + B$ for all $n \in \mathbb{N}$. Therefore

$$A_{m+n} = A_{m+n} \cap (A_{m+n} + B) = A_{m+n} \cap (A_m + B) =$$
$$A_m + (A_{m+n} \cap B) = A_m + (A_m \cap B) = A_m$$

for all $n \in \mathbb{N}$ and then the result follows.

(4) We proceed by induction on n. Assume that $B = A_1 + \ldots + A_{n-1}$ is notherian. By (1) $A/B = (A_n + B)/B \cong A_n/(A_n \cap B)$ is noetherian. By (2) A is noetherian.

(5) Suppose that A is noetherian, and let B be a non-zero submodule of A. Pick $0 \neq b_1 \in B$ and put $B_1 = b_1R$. If $B_1 \neq B$, then pick $0 \neq b_2 \in B \setminus B_1$ and put $B_2 = b_1R + b_2R$. Since $b_1 \notin B_1$, we have $B_2 \neq B_1$. Proceeding in this way, we construct an ascending chain of finitely generated submodules of A

$$B_1 \leq B_2 \leq \ldots \leq B_n \leq \ldots$$

such that every B_n is finitely generated. By the hypothesis, there is some $k \in \mathbb{N}$ such that $B_k = B_{k+n}$ for all $n \in \mathbb{N}$. It follows that $B = B_k$ and thus B is finitely generated.

Conversely, let

$$A_1 \leq A_2 \leq \ldots \leq A_n \leq \ldots$$

be an ascending chain of submodules. If $B = \bigcup_{n \in \mathbb{N}} A_n$, then there exists $b_1, \ldots, b_m \in A$ such that $B = b_1R + \ldots + b_mR$. Choose some r such that $b_1, \ldots, b_m \in A_r$. Clearly we have $B = A_r$, and then $A_r = A_n$ for all $n \geq r$, so that A is noetherian.

(6) is a direct consequence of (5).

(7) Let $A = a_1R + \ldots + a_tR$. Given $1 \leq i \leq t$ since $a_iR \cong R_R/Ann_R(a_i)$, each a_iR is noetherian by (1). By (4), A is also noetherian.

1.4. Theorem. *Let D be a Dedekind domain. Then*
(1) D is a noetherian ring.
(2) Every non – zero prime ideal of D is maximal.

Proof. Let A be any non – zero ideal of D. The equation $A\,A^{-1} = D$ implies the existence of elements $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_m \in A^{-1}$ such that $a_1 b_1 + \ldots + a_m b_m = 1$. If $x \in A$, then

$$x = x \cdot 1 = x\,(a_1 b_1 + \ldots + a_m b_m) = (x\,b_1)a_1 + \ldots + (xb_m)a_m.$$

Since $x\,b_1, \ldots, xb_m \in D$, an ideal A is generated by elements $a_1, \ldots, a_m$. Proposition 1.3 (6) implies that the ring D is noetherian.

Let P be a non – zero prime ideal of D. Choose a maximal ideal Q which includes P. We have $PQ^{-1} = QQ^{-1} = D$ and so PQ^{-1} is an ideal of D. Since P is a prime ideal, from the equation

$$(PQ^{-1})Q = P(Q^{-1}Q) = PD = P$$

we obtain that either $PQ^{-1} \leq P$ or $Q \leq P$. The first possibility implies

$$Q^{-1} \leq P^{-1}\,P\,Q^{-1} \leq P^{-1}\,P = D.$$

Thus $Q^{-1} = D$ and $Q = D$. Contradiction. This proves that $Q \leq P$, that is $P = Q$ is a maximal ideal of D.

To obtain further results on Dedekind domains we need in the following conception of integrality.

Let R be an integral domain, and let L be a field including R. There is no need for L to coincide with the field of fractions of R. An element $c \in L$ is said to be *integral over R*, or shortly *R- integral*, if

$$a_0 + a_1c + \ldots + a_{n-1}c^{n-1} + c^n = 0$$

for some elements $a_0, a_1, \ldots, a_n \in R$.

In other words, c is a root of a polynomial, all coefficients of which belong to R and the leading coefficient is identity.

We may define the R–integrality in other equivalent way, which is sometimes more suitable for applications. This is the content of

1.5. Proposition. *Let R be an integral domain, L a field, including R, c an element of L. The following assertions are equivalent:*
(1) c is integral over R.
(2) The subring $R[c]$ generated by R and x is finitely generated as R–module.

(3) L includes a finitely generated and non – zero R – submodule M such that $cM \leq M$.

Proof. (1) ⇒ (2). Since c is integral over R, there are the elements $a_0, a_1, \ldots, a_n \in R$ with the property $a_0 + a_1c + \ldots + a_{n-1}c_{n-1} + c_n = 0$. It follows that the elements $1, c, \ldots, c_{n-1}$ generate $R[c]$.

(2) ⇒ (3). The R – submodule $R[c]$ may serve for M.

(3) ⇒ (1). Let $cM \leq M$ for some finitely generated R – submodule M. Let $M = b_1M + \ldots + b_kM$. We have

$$cb_j = \sum_{1 \leq s \leq k} \alpha_{js} b_s, \quad 1 \leq j \leq k,$$

for some elements $\alpha_{js} \in R$, $1 \leq j, s \leq n$. Thus, we obtain the system of equations

$$\begin{aligned} (\alpha_{11} - c)b_1 &+ \alpha_{12}b_2 + \ldots + \alpha_{1k}b_k = 0 \\ \alpha_{21}b_1 &+ (\alpha_{22} - c)b_2 + \ldots + \alpha_{2k} = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{k1}b_1 &+ \alpha_{k2}b_2 + \ldots + (\alpha_{kk} - c)b_k = 0. \end{aligned}$$

Since this system has solutions, $f(c) = det\,(\|\alpha_{ij}\| - cE) = 0$. Obviously, $f(x)$ is a polynomial over R, and the leading coefficient of $f(x)$ is identity. This means that c is an integral over R.

1.6. Corollary. *Let L be a field, R a subring of L and* $1 \in R$. *The set C of all elements of L which are integral over R form a subring.*

Proof. Let a, c be R –integral elements of L. By Proposition 1.5 there are finitely generated R–submodules M_1 and M_2 such that $aM_1 \leq M_1$, $cM_2 \leq M_2$. Let

$$M_1 = b_1R + \ldots + b_kR, \; M_2 = d_1R + \ldots + d_rR.$$

Obviously,

$$M_1M_2 = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq r} b_i d_j R.$$

In particular, M_1M_2 is a finitely generated R – submodule. Let $x \in M_1M_2$, then

$$x = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq r} b_i d_j y_{ij}$$

for some $y_{ij} \in R$. Now

$$(a - c)x = ax - cx =$$

$$\Sigma_{1\le j\le r}\,(\Sigma_{1\le i\le k} ab_i)d_j y_{ij} + \Sigma_{1\le i\le k}\,(\Sigma_{1\le j\le r} cd_j)b_i y_{ij} \in M_1M_2;$$

$$(ac)x = ac\,(\Sigma_{1\le i\le k}\Sigma_{1\le j\le r} b_i d_j y_{ij}) =$$
$$a\,\Sigma_{1\le i\le k}\,(\Sigma_{1\le j\le r} cd_j)\,b_i y_{ij} = (\Sigma_{1\le i\le k}\,(\Sigma_{1\le j\le r} cd_j)\,ab_i)\,y_{ij} \in M_1M_2.$$

Proposition 1.5 implies that $a - c,\ ac \in C$. So C is a subring of L.

Let L be a field, R be a subring of L, and $1 \in R$. The subring of L consisting of all elements which are integrals over R is called *the integral closure of R in L*. If R is equal to its integral closure in its field of fractions, then we call it *integrally closed.*

Now we shall prove a theorem on the transitivity of integral closure.

1.7. Proposition. *Let L be a field, K a subfield of L, R a subring of K and $1 \in R$. If S is the integral closure of R in K, then the integral closures of R and S in L coincide.*

Proof. Let x be a S – integral element of L, that is,

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$$

for some elements $a_0, \ldots, a_{n-1} \in S$. The subring $R_1 = R[a_0, \ldots, a_{n-1}]$ is a finitely generated as an R – module, as can be seen from the consideration of the chain

$$R \le R[a_0] \le \ldots \le R[a_0, \ldots, a_{n-1}],$$

in which every ring is a finitely generated module over its predecessor. Moreover, x is R_1 – integral. Hence, by Proposition 1.5, $R_1[x]$ is a finitely generated R_1 – module. It follows that $R_1[x]$ is a finitely generated R–module. Since it is obviously non – zero and $xR_1[x] \le R_1[x]$, Proposition 1.5 implies R – integrality of x.

1.8. Corollary. *Let L be a field, R a subring of L and $1 \in R$. Then the integral closure of R in L is integrally closed.*

For the proof it suffices to apply Proposition 1.7 for $K = L$.

We shall now use the concept of integrality in order to characterize Dedekind domains. But first we need the following lemmas.

1.9. Lemma. *Let R be a noetherian integral domain, L its non - zero ideal. If $L \ne R$, then $R_1 \ldots P_m \le L \le P_1 \cap \ldots \cap P_m$ for some $< 0 > \ne P_1, \ldots, P_m \in Spec(R)$.*

Proof. Suppose that this statement is false. Then the family ✿ of the proper non – zero ideals that have no this property, is not empty. Let M be a maximal element of ✿. This ideal cannot be prime, since in this case we could take $j = 1$, $P_1 = M$. Hence there exist elements $a,\ b \notin M$ such that $ab \in M$. Put $A = M + aR$, $B = M + bR$. Then $AB \le M \le A \cap B$. If we assume that $A = R$, then $AB = B$, and this means that $B = M$.

Similarly $B \neq R$. Hence $A, B \notin \mathfrak{M}$, and this implies that M has it as well. This contradiction proves the lemma.

1.10. Lemma. *Let R be an integral domain satisfying the following conditions:*
(1) R is noetherian;
(2) every non – zero prime ideal of R is maximal;
(3) R is integrally closed.
Then every non – zero prime ideal of R is invertible.

Proof. Let $< 0 > \neq P \in Spec(R)$. Choose $0 \neq a \in P$ so that the principal ideal aR includes the product $P_1 \ldots P_k$ of the least possible number of prime ideals. Lemma 1.9 yields that a such choice is possible. Now P includes one of the ideals P_i, say P_1, and so, by (2), $P = P_1$. The ideal $a R$ does not include $P_2 \ldots P_k$, and so we may find $b \in P_2 \ldots P_k \setminus aR$. For this element b we have $bP \leq P_1 \ldots P_k \leq aR$ and $ba^{-1}P \leq R$, i.e. $ba^{-1} \in P' \setminus R$. This shows that $R \leq P'$ and $R \neq P'$.

Further, PP' is an ideal and, moreover, $P = RP \leq PP \leq R$. If we suppose that $PP' \neq R$, then must be $P P' = P$, and consequently $P(P')^n = P$ for all $n \in \mathbb{N}$. For every $0 \neq x \in P$, $y \in P' \setminus R$ this implies the inclusion $xy^n \in P$ for all $n \in \mathbb{N}$. In turn, this implies $xR[y] \leq R$, and so $xR[y]$ has to be an ideal of R. By (1) R has finitely many generators, say $u_1, \ldots, u_t$, and so the R – module $R[y]$ generates by the elements $x^{-1}u_1, \ldots, x^{-1}u_t$. Proposition 1.5 implies the y is integral over R, and condition (3) implies $y \in R$, what contradicts the selection of y.

1.11. Lemma. *Let R be an integral domain satisfying the following conditions:*
(1) R is noetherian;
(2) every non – zero prime ideal of R is maximal;
(3) R is integrally closed.
If L is a proper non – zero ideal of R, then $L = P_1 \ldots P_m$ where $< 0 > \neq P_j \in Spec(R)$, $1 \leq j \leq m$.

Proof. Assume that the lemma is false and choose a non – zero ideal $L \neq R$ which is not a product of prime ideals. By Lemma 1.9

$$P_1 \ldots P_m \leq L \quad \text{for some non – zero ideals } P_1, \ldots, P_m \in Spec(R).$$

We may assume that L is chosen so that the number m is the least possible. If $m = 1$, then (2) shows that L is itself a prime ideal. Hence $m \geq 2$. Choose a maximal ideal P including L. Then $P_1 \ldots P_m \leq P$, and it follows that $P_j = P$. We can assume that $P_1 = P$. Then

$$P_2 \ldots P_r \leq P^{-1}L \leq P^{-1}P = R,$$

whence $P^{-1}L$ is an ideal of R. By the selection of L we have

$$P^{-1}L = Q_1 \ldots Q_t \text{ where } Q_1, \ldots, Q_t \in Spec(R)$$

and finally, $L = PQ_1 \ldots, Q_t$. Contradiction.

1.12. Theorem. *Let D be an integral domain. Then D is a Dedekind domain if and only if it satisfies the following conditions:*

(1) D is noetjerian;

(2) every non – zero prime ideal of D is maximal;

(3) D is integrally closed.

Proof. Necessity. The necessity of (1) and (2) is contained in Theorem 1.4. To prove (3) take an element x of the field K of fractions of D, which is integral over D. By Proposition 1.5 $D[x]$ is a finitely generated D – module. Let $a_1, \ldots, a_m$ be its generators, and choose $0 \neq b \in D$ so that $ba_j \in D$ for each j, $1 \leq j \leq m$. Then $bD[x] \leq D$ and $D[x]$ is a fractional ideal. Moreover, $(D[x])^2 \leq D[x]$ ($D[x]$ is a subring), and we obtain

$$D[x] = DD[x] = (D[x])^{-1} D[x] D[x] = D,$$

which implies $x \in D$. This shows that D is integrally closed.

Sufficiency. Let L be a non – zero fractional ideal of R and choose $0 \neq a \in R$ such that $aL \leq R$. Since aI is an ideal of R, we may apply Lemma 1.11 to get $aL = P_1 \ldots P_m$ where some $< 0 > \neq P_1, \ldots, P_m \in Spec(R)$. This implies

$$L = a^{-1} P_1 \ldots P_m = (a^{-1} R) P_1 \ldots P_m.$$

By Proposition 1.2 and Lemma 1.10, L is a product of invertible ideals, and so it must be invertible itself.

Theorem 1.12 and Lemma 1.11 imply the following

1.13. Corollary. *Every proper non – zero ideal of a Dedekind domain can be represented as a product of prime ideals.*

Moreover,

1.14. Theorem *Every proper non – zero ideal of a Dedekind domain D can be represented uniquely as a product of non – zero prime ideals (products differing in order not being regarded as distinct).*

Proof. Only the uniqueness remains to be proved. Let L be a proper non – zero ideal of D, having at least two different representations as a product of prime ideals, say,

$$L = P_1 \ldots P_m = Q_1 \ldots Q_t \text{ where } P_1, \ldots, P_m, Q_1, \ldots, Q_t \in Spec(R).$$

We may assume that L is chosen so that the number m of prime ideal factors in one of the decompositions is the least possible. Now $Q_1, \ldots, Q_t \leq P_1$ whence P_1 includes one of the ideals $Q_1, \ldots, Q_t$. Let $Q_1 \leq P_1$. By Theorem 1.12, $Q_1 = P_1$. Using Lemma 1.10, we obtain

that $P_2 \ldots P_m = Q_2 \ldots Q_l$ are two different factorizations of L into prime ideals, and the number of factors in the first of them is less than m. This contradicts the choice L.

1.15. Corollary. *Let D be a Dedekind domain. Then the group $\mathcal{FI}(D)$ is a free abelian group generated by the non – zero prime ideals of D.*

Proof. Let A be a fractional ideal of R and choose $0 \neq a \in R$ so that $aA \leq R$. Clearly aA is an ideal of R. Hence $A = (aR)^{-1}(aA)$ is a product of powers of prime ideals with integral, not necessarily positive, exponents. Hence the prime ideals generate the group $\mathcal{FI}(D)$, and Theorem 1.14 shows that they generate it freely.

Let D be a Dedekind domain, A and B be ideals of D. We say that A *is divisible by* B, or that B *divides* A, if $A = BC$ for some ideal C of D.

1.16. Corollary. *Let D be a Dedekind domain. Then every non – zero ideal can be divisible only by finitely many distinct ideals.*

Thus we see that every fractional ideal A of a Dedekind domain D may be uniquely written in the form

$$A = P_1^{a(P_1)} \ldots P_r^{a(P_r)} \text{ where } P_1, \ldots, P_r \in Spec(D),\ a(P_1), \ldots, a(P_r) \in \mathbb{Z}.$$

It is convenient to use the following general form of this notation:

$$A = \prod_{P \in Spec(D)} P^{a(P)}.$$

Of course, using this form, we will assume, that only a finitely many of $a(P)$ are non –zero.

For the ideals of R we can regain many results of elementary number theory connected with the notions of the least common multiple or the greatest common divisor.

Let D be a Dedekind domain, A and B be ideals of D. The *greatest common divisor of A and B* is the ideal $GCD(A, B)$ with the following properties:
$GCD(A,B)$ divides A and B;
if C is an ideal of D such that C divides A and B, then C divides $GCD(A, B)$.

The existence and uniqueness of such an ideal follows from the product representation. It should only be noted that if

$$A = \prod_{P \in (Spec(D) \setminus <0>)} P^{a(P)},\ B = \prod_{P \in (Spec(D) \setminus <0>)} P^{b(P)}$$

are ideals, then $a(P)$, $b(P)$ are non – negative, and

A divides B if and only if $a(P) \leq b(P)$ for all $P \in Spec(D)$.

This clearly implies that

$$GCD(A, B) = \prod_{P \in (Spec(D) \setminus <0>)} P^{c(P)} \text{ where } c(P) = min\{a(P), b(P)\}.$$

Similarly, the *least common multiple of A and B* is the ideal $LCM(A, B)$ with the following properties:

A and B divide LCM(A, B);

if C is an ideal of D such that A and B divide C, then LCM(A, B) divides C.

It is easy to see that

$$LCM(A, B) = \prod_{P \in (Spec(D) \setminus <0>)} P^{d(P)} \text{ where } d(P) = max\{a(P), b(P)\}.$$

Let R be a commutative ring, A and B be ideals of R. We say that A, B are *relatively prime*, if $A + B = R$.

Now we obtain some other accustomed properties.

1.17. Proposition. *Let D be a Dedekind domain. Then the following assertion hold.*

(1) If A, B are fractional ideals, then B divides A if and only if $A \leq B$.

(2) If A is a fractional ideal, then there exists a principal fractional ideal aD such that D includes $(aD)A^{-1}$.

(3) If A and B are ideals of R, then $GCD(A, B) = A + B$, $LCM(A, B) = A \cap B$.

(4) If A and B are ideals of R, then $AB = (A + B)(A \cap B)$.

(5) If A and B are relatively prime ideals of R, then $AB = A \cap B$.

Proof. The equation $A = BC$ implies that $A \leq B$. If $A \leq B$, then

$$AB^{-1} \leq BB^{-1} = D,$$

so that $A = BC$ where $C = AB^{-1}$.

Let $0 \neq a \in A$, then $aD \leq D$ and we may apply (1) to prove (2).

(1) shows that both ideals A and B are divisible by $A + B$, and if an ideal C divides A and B, then $A \leq C$ and $B \leq C$. Then $A + B \leq C$. It follows that $A + B = GCD(A, B)$.

Using the similar arguments we can prove that $LCM(A, B) = A \cap B$.

Let

$$A = \prod_{P \in (Spec(D) \setminus <0>)} P^{a(P)}, B = \prod_{P \in (Spec(D) \setminus <0>)} P^{b(P)},$$

then

$$AB = \prod_{P \in (Spec(D) \setminus <0>)} P^{c(P) + d(P)}$$

where

$$c(P) = min\{a(P), b(P)\}\ d(P) = max\{a(P), b(P)\}.$$

It follows that

$$AB = GCD(A, B)\ LCM(A, B) = (A + B)(A \cap B) \text{ by (3).}$$

Finally, if A and B are relatively prime, then by (4)

$$AB = (A + B)(A \cap B) = D(A \cap B) = A \cap B.$$

1.18. Lemma. *Let R be a commutative ring, A and B be ideals of R. If A, B are relatively prime then A^k and B^m are relatively prime for every $k, m \in \mathbb{N}$.*

Proof. From $A + B = R$ we obtain that $1 = a + b$ for some $a \in A$, $b \in B$. Then

$$1 = 1^{k+m} = (a + b)^{k+m} =$$
$$a^{k+m} + \binom{k+m}{1} a^{k+m-1} b + \ldots + \binom{k+m}{k+m-1} ab^{k+m-1} + b^{k+m}.$$

Let $s, t \in \mathbb{N}$ and $k + m = s + t$. If $s \geq k$, then $a^s \in A^k$ and $a^s b^t \in A^k$ because A is an ideal. Hence every summand belongs to A^k or B^m. This means that $1 \in A^k + B^l$.

1.19. Lemma. *Let R be a commutative ring, $A_1, \ldots, A_m$ be ideals of R. The ideals A_i, A_j are relatively prime for each $i \neq j$ if and only if*

$$A_j + A_1 \ldots A_{j-1} A_{j+1} \ldots A_m = R$$

for every j, $1 \leq j \leq m$.

Proof. Since $A_j + A_1 = R$, $1 = a_j + b_j$ for some elements $a_1 \in A_1$, $b_j \in A_j$, $2 \leq j \leq m$. Then

$$1 = (a_1 + b_1) \ldots (a_{m-1} + b_{m-1}).$$

Every summand from the decomposition of the right side, except the last, contains an element a_i, therefore it belongs to A_1. The last summand $b_1 \ldots b_{m-1}$ belongs to $A_2 \ldots A_m$. Hence $R = A_1 + A_2 \ldots A_{m-1}$. Similarly we can justify the equation

$$R = A_i + A_1 \ldots A_{j-1} A_{j+1} \ldots A_m$$

for every j, $1 \leq j \leq m$.

Conversely, *let* $R = A_j + A_1 \ldots A_{j-1} A_{j+1} \ldots A_m$ for every j, $1 \leq j \leq m$. Since $A_1 \ldots A_{j-1} A_{j+1} \ldots A_m \leq A_k$ for each $j \neq k$, it implies $R = A_k + A_j$ for $j \neq k$.

1.20. Corollary *Let D be a Dedekind domain, $A_1, \ldots, A_m$ be non – zero ideals of D. If A_i and A_j are relatively prime for every $i \neq j$, then*

$$A_1 \ldots A_m = A_1 \cap \ldots \cap A_m.$$

Proof. We will use induction on m. If $m = 2$, then the statement follows from Lemma 1.18. Let $m > 2$ and let we have already proved the equation

$$C = A_1 \ldots A_{m-1} = A_1 \cap \ldots \cap A_{m-1}.$$

Lemma 1.19 yields that $A_r + C = R$. Proposition 1.17 implies that

$$A_1 \ldots A_{m-1}A_m = C A_m = (C + A_m)(C \cap A_m) =$$
$$D(C \cap A_m) = C \cap A_m = A_1 \cap \ldots \cap A_{m-1} \cap A_m.$$

1.21. Theorem (Chinese Remainder Theorem). *Let D be a Dedekind domain, $A_1, \ldots, A_k$ be non – zero ideals of D, $a_1, \ldots, a_k \in D$. If A_i and A_j are relatively prime for every pair $i \neq j$, then there exists an element $a \in D$ such that*

$$a_i + A_i = a + A_i \text{ for each } i,\ 1 \leq i \leq k.$$

If b is an element such that

$$b + A_i = a_i + A_i \text{ for each } i,\ 1 \leq i \leq k,$$

then $a + A_1 \ldots A_k = b + A_1 \ldots A_k$.

Proof. Lemma 1.19 yields that

$$1 = b_i + c_i, \text{ where } b_i \in A_i,\ c_i \in A_1 \ldots A_{i-1}A_{i+1} \ldots A_k,\ 1 \leq i \leq k.$$

Put

$$a = a_1 c_1 + \ldots + a_k c_k.$$

We have

$$a + A_i = a_i c_i + A_i,\ 1 \leq i \leq k.$$

Since $1 = b_i + c_i$, we have $c_i + A_i = 1 + A_i$ and

$$a + A_i = a_i c_i + A_i = (a_i + A_i)(c_i + A_i) = a_i + A_i.$$

Let b be an element with the following property:

$$b + A_i = a_i + A_i \text{ for each } i,\ 1 \leq i \leq k.$$

Then $a - b \in A_i$ for every $i,\ 1 \leq i \leq k$, that is, $a - b \in A_1 \cap \ldots \cap A_k$. Corollary 1.20 implies that $A_1 \cap \ldots \cap A_k = A_1 \ldots A_k$. Hence $a - b \in A_1 \ldots A_k$.

1.22. Corollary. *Let D be a Dedekind domain, $A_1, \ldots, A_k$ be non – zero ideals of D. If A_i and A_j are relatively prime for each $i \neq j$ then*

$$D/A_1 \ldots A_k \cong D/A_1 \times \ldots \times D/A_k.$$

Proof. Consider the mapping

$$\phi: D \longrightarrow D/A_1 \times \ldots \times D/A_k$$

which we define by the rule

$$a\phi = (a + A_1, \ldots, a + A_k),\ a \in D.$$

It is easy to see that ϕ is a homomorphism. Obviously, $Ker\ \phi = A_1 \cap \ldots \cap A_k$. Corollary 1.20 shows that $Ker\ \phi = A_1 \ldots A_k$, and Theorem 1.21 implies that

$$Im\ \phi = D/A_1 \times \ldots \times D/A_k.$$

1.23. Corollary. *Let D be a Dedekind domain, I be a non – zero ideal of D, $I = P_1^{m_1} \ldots P_k^{m_k}$ where $P_1, \ldots, P_k$ are prime ideals of D, $P_i \neq P_j$ for $i \neq j$, and $m_i > 0$ for each i. Then $D/I \cong D/P_1^{m_1} \times \ldots \times D/P_k^{m_k}$.*

In fact, Lemma 1.18 implies that $P_i^{m_i}$ and $P_j^{m_j}$ are relatively prime for $i \neq j$. Now we can use Corollary 1.19.

We have already mentioned that every principal ideal domain is a Dedekind domain. The following two theorems show that Dedekind domains are near to principal ideal domains.

1.24. Proposition *Let D be a Dedekind domain, Q a prime non – zero ideal of D, $m \in \mathbb{N}$, $y \in Q \setminus Q^2$. If B is an ideal such that $B \geq P^m$, then $B = Q^k$ for some $k \leq m$. If $y \in Q \setminus Q^2$, then $Q^k = y^k D + Q^m$ for $k \leq m$.*

Proof. By Corollary 1.13 $B = \Pi_{P \in (Spec(D) \setminus <0>)} P^{a(P)}$. In other words, $B = Q^k A$. If we suppose that $k \geq m$, then $B = Q^k A \leq Q^k$ by Proposition 1.15. Thus,

$$B \leq Q^k = Q^{k-m} Q^m \leq Q^m,$$

that is, $B = Q^m$. So $k \leq m$. Suppose that $A \neq R$, then Q and A are relatively prime by Lemmas 1.18 and 1.19. We have $BA^{-1} \leq Q^k \leq R$, whence $A^{-1} \leq B^{-1}$ and $B \leq A$. It follows that $Q^m \leq A$ By Lemma 1.18, $A + Q^m = R$, what implies $A = R$. This contradiction shows that $B = Q^k$.

Since $y \in Q$, $yD \leq Q$. Taking in account that $Q^m \leq Q$, we obtain $yD + Q^m \leq Q$. By the above $yD + Q^m = Q^s$ for some $s \leq m$. If we suppose that $s > 1$, then

$$yD + Q^m \leq Q^s = Q^2\, Q^{s-2} \leq Q^2.$$

that is, $yD \leq Q^2$, what contradicts to the choce of y. This contradiction implies $s = 1$, that is, $yD + Q^m = Q$. Let $x_1, x_2 \in Q$, then $x_j = yz_j + u_j$ where $z_j \in D$, $u_j \in Q^m$, $j \in \{1, 2\}$. We have

$$x_1 x_2 = (yz_1 + u_1)(yz_2 + u_2) = y^2 z_1 z_2 + yz_1 u_2 + yz_2 u_1 + u_1 u_2 \in y^2 D + Q^m.$$

It follows that $Q^2 \leq y^2 D + Q^m$, and hence $Q^2 = y^2 D + Q^m$. With the help of ordinary induction we obtain the equation $Q^k = y^k D + Q^m$, for every $k \leq m$.

1.25. Corollary *Let D be a Dedekind domain, A a non – zero ideal. Then every ideal of factor- ring D/A is principal.*

Proof. Let B be an ideal including A. By Corollary 1.13, $A = P_1^{k_1} \ldots P_t^{k_t}$ for some $P_1, \ldots, P_t \in Spec(D)$. Corollary 1.23 implies the isomorphism

$$D/A \cong D/P_1^{k_1} \times \ldots \times D/P_t^{k_t}.$$

By Proposition 1.24, there are elements $a_1, \ldots, a_t$ such that

$$B + P_j^{k_j} = a_j D + P_j^{k_j}, 1 \leq j \leq t.$$

Using Theorem 1.21, we can find an element a such that

$$a + P_j^{k_j} = aj + P_j^{k_j}, 1 \leq j \leq t.$$

Now it is not hard to prove that $B = aD + A$.

1.26. Corollary. *Let D be a Dedekind domain, A be a non – zero ideal of D. There are elements a_1 and a_2 such that $A = a_1 D + a_2 D$.*

Proof. Let $0 \neq a_1 \in A$ and $B = a_1 D$. Corollary 1.25 implies that

$$A/B = (a_2 + B)D = a_2 D + B/B$$

for some element $a_2 \in A$. Then $A = a_2 D + a_1 D$.

1.27. Corollary *Let D be a Dedekind domain, A and B be non – zero ideals of D. There exists a non – zero ideal C such that $B + C = D$, and AC is a principal ideal.*

Proof. Since D does not contain the zero – divisors, $AB \neq < 0 >$. Corollary 1.25 implies that $A/AB = aD + AB/AB$ for some element $a \in A$. The inclusion $aD \leq A$ together with Proposition 1.17 prove that $aD = AC$ for some ideal $C \neq < 0 >$. Thus $AC + AB = A$. Multiplying by A^{-1} we have $C + B = D$.

1.28. Corollary. *Let D be a Dedekind domain, A and B be non – zero ideals of D. Then the D – modules D/A and B/AB are isomorphic. In particular, the D – modules D/A and A^k/A^{k+1} are isomorphic for each $k \in \mathbb{N}$.*

Proof. By Corollary 1.27, there exists a non – zero ideal C such that $BC = aD$ for some $a \in D$ and $C + A = D$. Define the mapping $\phi: D \longrightarrow B/AB$ by the rule $x\phi = ax + AB, x \in D$. We have

$$(x + y)\phi = a(x + y) + AB = ax + ay + AB = (ax + AB) + (ay + AB) = x\phi + y\phi;$$
$$(dx)\phi = a(dx) + AB = (ax)d + AB = (ax + AB)d = (x\phi)d$$

for every $d \in D$. Furthermore

$$aD + AB = CB + AB = (C + A)B = DB = B.$$

So $Im\ \phi = B/AB$. If $x \in Ker\ \phi$, then $ax \in AB$ and $axC \leq ABC = aA$. It follows that $xC \leq A$. From the equation $A + C = D$, we obtain that $1 = a_1 + c$ for some elements $a_1 \in A$, $c \in C$, and $x = x \cdot 1 = xa_1 + xc \in A$. Conversely, if $x \in A$, then $ax \in AB$, that is $x\phi = ax + AB = AB$. So $Ker\ \phi = A$.

1.29. Theorem. *Let D be a Dedekind domain. If the set $Spec(D)$ is finite, then D is a principal ideal domain.*

Proof. Let Q be a prime ideal, put $L = \bigcap_{n \in N} Q^n$ and suppose that $L \neq < 0 >$. By Corollary 6.13 $L = \prod_{P \in (Spec(D) \setminus < 0 >)} P^{a(P)}$. In other words, $L = Q^k A$. Suppose that $A \neq R$; then Q and A are relatively prime by Lemmas 6.18 and 6.19. If $k = 0$, then $A \leq Q^n$ for each $n \in \mathbb{N}$, and the equation $R = A + Q$ implies $R = Q$. Hence $k \neq 0$. In this case we have $Q^k A \leq Q^{k+1}$ what implies $A \leq Q$. This contradiction shows that $\bigcap_{n \in N} Q^n = < 0 >$. If we assume that $Q = Q^2$, then $Q = Q^n$ for all $n \in \mathbb{N}$. But in this case $\bigcap_{n \in N} Q^n = Q$. This contradiction proves that $Q \neq Q^2$ for every prime ideal Q.

Let

$$Spec(D) \setminus < 0 > = \{ P_1, \ldots, P_m \}.$$

We choose elements $b_j \in P_j \setminus P_j^2$, $1 \leq j \leq m$. By Theorem 1.21, there exists an element a_j such that $a_j + P_j^2 = b_j + P_j^2$, $a_j + P_s = 1 + P_s$ for $s \neq j$. Consider the ideal $a_j D$. Corollary 6.13 implies that

$$a_j D = P_1^{k_{j1}} \ldots P_m^{k_{jm}} \text{ where } k_{js} \geq 0,\ 1 \leq j, s \leq m.$$

If $k_{js} \neq 0$ for some $s \neq j$, then $a_j D \leq P_s$. But $a_j + P_s = 1 + P_s \neq P_s$. It shows that $k_{js} = 0$ for all $s \neq j$, that is, $a_j D = P_j^{k_{jj}}$. If we suppose that $k_{jj} \geq 2$, then $a_j \in P_j^2$ and this is impossible. So $k_{jj} = 1$, what means that $a_j D = P_j$. In other words, every prime ideal is principal. Corollary 6.13 yields that every non – zero ideal is principal.

Finally, we present the following theorem, which shows how broad the class of Dedekind domains is.

1.30. Theorem. *Let R be a noetherian integral domain in which every non – zero prime ideal is maximal, F be the field of fractions of R, K be a finite field extension of F. Then the integral closure of R in K is a Dedekind domain.*

The proof of this theorem can be found in the book [BN1961B, ch. VII, 2, Corollary 2 of Proposition 5]. It requires implementation of some specific results from the field extension theory; therefore we omit it.

Chapter 2

Principal Ideal Domains

In this chapter we consider one especially important for applications specific type of the Dedekind domains.

We begin with an ordinary concept of *divisibility* in the rings. Let R be a commutative ring. We say that an element $a \in R$ *is divisible by an element* b, or that b *divides* a, if $a = bc$ for some element $c \in R$. In this case, we will say also that b *is a divisor of* a. We will denote this by $a \mid b$.

Let $a \mid b$ and $b \mid a$, that is, $b = ac$ and $a = bd$ for suitable elements $c, d \in R$. Then $a = a(cd)$, and $cd = 1$ or $c, d \in \mathbf{U}(R)$.

The elements $a, b \in R$ are said to be *associated*, if $b = au$ for some element $u \in \mathbf{U}(R)$.

It is almost obvious that the relation "to be the associated elements" is an equivalence relation. The class of equivalence allied with this relation is called the *associated class* of a ring R.

If $u \in \mathbf{U}(R)$, then the element u is a divisor of each element $a \in R$. The following equations proves this fact

$$a = ae = a(u^{-1}u) = (au^{-1})u.$$

We say that an element c is a *non ~ proper divisor* of an element a, if either $c \in \mathbf{U}(R)$ or c is associated with a.

Dually, if $a = bc$ where $b, c \notin \mathbf{U}(R)$, then the elements b and c are called the *proper divisors* of element a.

Observe the following elementary properties of divisibility.

2.1. Proposition. *Let R be an integral domain.*

(1) If an element a of the ring R divides an element b and b divides c, then a divides c.

(2) If an element a of the ring R divides the elements b, c, then a divides $b + c$.

(3) If an element a of the ring R divides an element b, then a divides bc.

(4) If an element a of the ring R divides elements $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ are arbitrary elements of R, then a divides $b_1c_1 + \ldots + b_n c_n$.

In Chapter 1 we considered the concept of divisibility for ideals. In the following way this concept is connected with the concept of divisibility of elements.

2.2. Proposition. *Let R be an integral domain. Then an element a of a ring R divides an element b if and only if $bR \leq aR$.*

Proof. If $a \mid b$, then $b = ac$ for some element $c \in R$. In particular, $b \in aR$ and therefore $bR \leq aR$. Conversely, if $bR \leq aR$, then $b \in aR$, and hence $b = ac$ for some element $c \in R$.

2.3. Corollary. *Let R be an integral domain.*

(1) The elements a, b of the ring R are associated if and only if $bR = aR$.

(2) If a is a proper divisor of an element b, then $bR \leq aR$ and $bR \neq aR$.

Let R be an integral domain and $a, b \in R$. An element d of the ring R is called a *greatest common divisor of a and b* if d satisfies the following conditions:

(GCD 1) *d divides a and b;*

(GCD 2) *if c is a common divisor of a, b, then c divides d.*

Clearly if d is a greatest common divisor of the elements a and b, then du satisfies the conditions **(GCD 1)** and **(GCD 2)** for each element $u \in \mathbf{U}(R)$. Conversely, if an element d_1 satisfies the conditions **(GCD 1)** and **(GCD 2)**, then the elements d, d_1 divide one to another, and thus, they are associated. Hence all greatest common divisors of the

elements a, b compound the class of associated elements. The fact that an element d is one of greatest common divisors of the elements a, b we will designate by the symbol $d = \mathbf{GCD}(a, b)$.

Dual to the concept of a greatest common divisor is a concept of a least common multiple.

Let R be an integral domain and $a, b \in R$. An element w of a ring R is called a *least common multiple of a and b* if w satisfies the following conditions:

(LCM 1) *a and b divide w;*

(LCM 2) *if a and b divide c, then w divides c.*

Clearly if w is a least common multiple of the elements a, b, then wu also satisfies the conditions **(LCM 1)** and **(LCM 2)** for each element $u \in \mathbf{U}(R)$. Conversely, if an element w_1 satisfies the conditions **(LCM 1)** and **(LCM 2)**, then the elements w, w_1 divide one to another, and thus, they are associated. Hence, all least common multiples of the elements a, b compound the class of associated elements. The fact that an element w is one of least common multiples of the elements a, b we will designate by the symbol $w = \mathbf{LCM}(a, b)$.

If R is an arbitrary commutative ring, then not each pair of its element has a greatest common divisor and a least common multiple. However, in the following kind of rings such elements always exist.

Recall that an integral domain R is called a *principal ideal domain*, if each ideal of R is principal. As we observed in Chapter 1, every principal ideal domain is a Dedekind domain.

2.4. Theorem. *Let R be a principal ideal domain. Then for each pair a, b there are their greatest common divisor and least common multiple. Furthermore, the equations $d = \mathbf{GCD}(a, b)$ and $dR = aR + bR$ (respectively $m = \mathbf{LCM}(a, b)$ and $mR = aR \cap bR$) are equivalent.*

Proof. Since every ideal of R is principal, there exists an element $d \in R$ such that $dR = aR + bR$. The inclusions $aR \leq dR$ and $bR \leq dR$ together with Proposition 2.2 shows that $d \mid a$ and $d \mid b$. Let c be a common divisor of the elements a and b. Using Proposition 2.2, we obtain the inclusions $aR \leq cR$ and $bR \leq cR$. This implies that $dR = aR + bR \leq cR$. In turn, it follows that c is a divisor of element d. Hence d satisfies the conditions **(GCD 1)** and **(GCD 2)**. Conversely, if d_1 is the greatest common divisor of elements a and b, then, as we saw above, the elements d, d_1 are associated. In particular, they generate the same ideal, that is, $d_1R = dR = aR + bR$.

Since every ideal is principal, there exists an element m such that $mR = aR \cap bR$. The inclusions $mR \leq aR$ and $mR \leq bR$ together with Proposition 2.2 show that $a \mid m$ and $b \mid m$. Let c be a common multiple of elements a, b. Using Proposition 2.2, we obtain the inclusions $cR \leq aR$ and $cR \leq bR$. This implies that $cR \leq aR \cap bR = mR$. In turn, it follows that m divides c. Thus, m satisfies the conditions **(LCM 1)** and **(LCM 2)**.

Conversely, if m_1 is the least common multiple of elements a and b, then, as saw above, the elements m and m_1 are associated. In particular, they generate the same ideal, that is, $m_1R = mR = aR \cap bR$.

2.5. Corollary. *Let R be a principal ideal domain and a, b be elements of R. Then the following assertions hold.*

(1) $a = \mathbf{GCD}(a, b)$ *if and only if* a *divides* b.

(2) $\mathbf{GCD}(a, 0) = a$.

(3) $\mathbf{GCD}(ax, bx) = x\,\mathbf{GCD}(a, b)$.

(4) $\mathbf{GCD}(\mathbf{GCD}(a, b), c) = \mathbf{GCD}(a, \mathbf{GCD}(b, c))$.

(5) $a = \mathbf{LCM}(a, b)$ *if and only if* b *divides* a.

(6) $\mathbf{LCM}(a, 0) = 0$.

(7) $\mathbf{LCM}(ax, bx) = x\,\mathbf{LCM}(a, b)$.

(8) $\mathbf{LCM}(\mathbf{LCM}(a, b), c) = \mathbf{LCM}(a, \mathbf{LCM}(b, c))$.

(9) If $d = \mathbf{GCD}(a, b)$, *then there are the elements* $x, y \in R$ *such that* $d = ax + by$.

(10) If a, b *are non – zero elements, then* $\mathbf{GCD}(a, b)$ *and* $\mathbf{LCM}(a, b)$ *are likewise non – zero.*

Proof. All assertions except the last one are almost obvious. We will prove only the last assertion. Let $d = \mathbf{GCD}(a, b)$, $m = \mathbf{LCM}(a, b)$. By Theorem 2.4, $dR = aR + bR$, $mR = aR \cap bR$. The inclusions $aR \leq dR$ and $bR \leq dR$ guarantee that the ideal dR is non – zero. Hence, the element d must be non – zero.

Suppose that $m = 0$. It follows that $aR \cap bR = \{0\}$. The element ab belongs to both ideals aR and bR, that is $ab \in aR \cap bR$. Then $ab = 0$. This is impossible, because R does not contain zero divisors. This contradiction shows that $m \neq 0$.

2.6. Corollary. *Let R be a principal ideal domain and a, b the elements of R. Then the following assertions hold:*

(1) $d = \mathbf{GCD}(a, b)$ *if and only if* $dR = \mathbf{GCD}(aR, bR)$.

(2) $m = \mathbf{LCM}(a, b)$ *if and only if* $mR = \mathbf{LCM}(aR, bR)$.

These assertions follow from Theorem 2.4 and Proposition 1.17.

The elements a and b of a ring R is said to be *relatively prime* if 1 is one of the greatest common divisors of these elements.

2.7. Corollary. *Let R be a principal ideal domain and a, b the elements of R. Then a, b are relatively prime if and only if there exist the elements* $x, y \in R$ *such that* $1 = ax + by$.

Proof. If a, b are relatively prime, then 1 is their greatest common divisor, and we can apply Corollary 2.5. Conversely, let x, y be the elements of R such that $1 = ax + by$. It follows that $1 \in aR + bR$. It follwos that $R = aR + bR$. Since $1R = R$, Theorem 2.4 impliess that $1 = \mathbf{GCD}(a, b)$.

2.8. Corollary. *Let R be a principal ideal domain and a, b be non – zero elements of R. Suppose that* $d = \mathbf{GCD}(a, b)$ *and* $a = da_1$, $b = db_1$. *Then the elements* a_1 *and* b_1 *are relatively prime.*

Proof. By Corollary 2.5, R contains elements x, y such that $d = ax + by$. We obtain now $d = ax + by = da_1x + db_1y$. Since R has no zero – divisors, it follows that $1 = a_1x + b_1y$. Corollary 2.7 implies that the elements a_1 and b_1 are relatively prime.

Note the following properties of relatively prime elements.

2.9. Proposition. *Let R be a principal ideal domain and a, b, and $c \in R$.*

(1) If a and b are relatively prime and a and c are relatively prime, then the elements a and bc are relatively prime.

(2) Suppose that a divides bc and a, b are relatively prime. Then a divides c.

Proof. (1) By Corollary 2.7, R contains such elements x_1, x_2, y_1, y_2 that $1 = ax_1 + bx_2$, $1 = ay_1 + cy_2$. We have now

$$1 = (ax_1 + bx_2)(ay_1 + cy_2) = a(ax_1y_1 + bx_2y_1 + cx_1y_2) + (bc)(x_2y_2).$$

Using again Corollary 2.7, we obtain that a and bc are relatively prime.

(2) By Corollary 2.7, R contains such elements x_1, x_2 that $1 = ax_1 + bx_2$. Then $c = c1 = cax_1 + cbx_2$. Since a divides bc, then a divides c.

The next assertion describes connections between greatest common divisors and least common multiples of two elements.

2.10. Theorem. *Let R be a principal ideal domain and a, b be non – zero elements of R.*

(1) If m is a least common multiple of the elements a, b and $ab = md$, then d is a greatest common divisor of a, b.

(2) If d is a greatest common divisor of a, b and $a = da_1$, $b = db_1$, then $m = da_1b_1$ is a least common multiple of the elements a, b.

Proof. (1) We have $m = a_2a$, $m = b_2b$ for certain elements $a_2, b_2 \in R$. Then $ab = md = aa_2d$. Since R has no zero divisors, it follows that $b = a_2d$. This means that d is a divisor of b. Similarly, d is a divisor of a. Suppose that c is an arbitrary common divisor of the elements a, b, that is, $a = a_3c$, $b = b_3c$ for some elements a_3, b_3. Clearly, a and b are the divisors of a_3b_3c. Then there exists an element $z \in R$ such that $a_3b_3c = mz$. Hence, $md = ab = a_3b_3cc = mzc$. By Corollary 2.5, $m \neq 0$; then we have $d = zc$. Consequently, d satisfies the conditions **(GCD 1)**, **(GCD 2)**.

(2) Obviously, a and b are divisors of m. Let c be a common multiple of a and b, that is, $c = aa_4$, $c = bb_4$ for some elements $a_4, b_4 \in R$. We have $c = aa_4 = da_1a_4$ and $c = bb_4 = db_1b_4$, so that, $da_1a_4 = db_1b_4$. By Corollary 2.5, $d \neq 0$; thus we obtain $a_1a_4 = b_1b_4$. Corollary 2.8 yields that the elements a_1 and b_1 are relatively prime. Proposition 2.9 implies that a_1 divides b_4, $b_4 = a_1x$. Thus $c = db_1b_4 = db_1a_1x = mx$. So, we can see that m satisfies conditions **(LCM 1)** and **(LCM 2)**.

A non – zero element x of an integral domain R is said to be a *prime* if an ideal xR is a prime ideal. Expressed otherwise, a non – zero element x of R is a prime if x is not non – invertible and satisfies the following condition.

If x divides ab, then either x divides a or x divides b.

Let x be a prime element and $x = ab$. Then by definition, $x \mid a$ or $x \mid b$, say $x \mid a$. It follows that $a = xz = abz$ for some element $z \in R$. Since $a \neq 0$, it follows that $bz = 1$, so $b \in \mathbf{U}(R)$. This means that x is not a product of two proper divisors. In other words, in every presentation x = ab at least one of elements a, b is invertible.

2.11. Proposition. *Let D be a Dedekind domain. Then the following assertions are equivalent:*

(1) an element x of a ring D is prime;

(2) an ideal xD is maximal.

Proof. By definition, the statement that x is a prime element means that an ideal xD is prime. Since x is non – zero, xD is a non – zero prime ideal. By Theorem 1.4, xD is maximal. Thus, (1) follows (2). Conversely, if an ideal xD is maximal, then D/xD is a field. In particular, it is integral domain. It follows that xD is a prime ideal.

2.12. Corollary. *Let R be a principal ideal domain. Then the following assertions are equivalent:*

(1) an element x of a ring R is prime;

(2) an ideal xR is maximal.

2.13. Lemma. *Let R be a principal ideal domain and x be a prime element of R. If a is an element of R then either x divides a or the elements a, x are relatively prime.*

Proof. Let $d = \mathrm{GCD}(a, x)$. Since d is a divisor of x, either elements x and d are associated or $d \in \mathbf{U}(R)$. In a first case, x is a greatest common divisor of x and a, in particular, x divides a. In the second case, elements x and a are relatively prime.

the following important properties of principal ideal domains we can obtain now.

2.14. Theorem *Let R be a principal ideal domain and a be a non – zero element of R.*

(1) If $a \notin \mathbf{U}(R)$, then a is a product of the prime elements of R.

(2) If $a \notin \mathbf{U}(R)$ and $a = x_1 \ldots x_n = y_1 \ldots y_m$ are two decompositions of a in a products of primes elements, then $n = m$ and we may renumber the members of the second decomposition in a such way that $q_j = u_j p_j$ for some $u_j \in \mathbf{U}(R)$, $1 \leq j \leq n$.

Proof. (1) Suppose the contrary, and denote by S a subset of such non – zero elements of $R \setminus \mathbf{U}(R)$ that does not decompose into a product of prime elements. By our assumption, $S \neq \varnothing$. Consider an ascending chain of ideals

$$z_1R \leq z_2R \leq \ldots \leq z_nR \leq \ldots$$

such that $z_n \in S$ for each $n \in \mathbf{N}$. Since $H = \cup_{n \in N} z_nR$ is an ideal, there is an element z such that $H = zR$. By the construction of H, there exists a number $k \in \mathbf{N}$, such that $z \in z_kR$. It follows that $zR \leq z_kR$. On the other hand, $z_kR \leq H = zR$, so that, $z_kR = zR$.

Let $a = a_1$ be an arbitrary element of S. Suppose first that for each ideal bR with the properties $aR \leq bR$ and $aR \neq bR$ an element b does not belong to S. If $b \in U(R)$, then $bR = R$. Assume that $b \notin U(R)$. By the choice of S, it shows that b is decomposed in a product of prime elements. By Proposition 2.2 b is a divisor of a, that is, $a = bc$ for some element $c \in R$. We have $aR \leq cR$. If we suppose that $aR = cR$, then $c = ua$ for some $u \in U(R)$. Therefore $a = bc = bua$, whence $b \in U(R)$. The later contradicts our assumption concerning b. This contradiction shows that $aR \neq cR$. If $c \in U(R)$, then $a = bc$ is a product of prime elements. If $c \notin U(R)$, then by $c \notin S$, c is a product of prime elements. But then, $a = bc$ is a product of prime elements. This contradiction shows that for each element $a = a_1$ of subset S there exists an element a_2 such that $a_2 \in S$, $a_1R \leq a_2R$ and $a_1R \neq a_2R$. However, this gives us a possibility to build an infinite ascending chain of ideals

$$a_1R \leq a_2R \leq \ldots \leq a_nR \leq \ldots$$

such that $a_n \in S$ for all $n \in \mathbb{N}$. We saw above that this is impossible. This contradiction implies that $S = \varnothing$. Thus (1) is proved.

(2) Put $P_j = x_jR$, $1 \leq j \leq n$, $Q_k = y_kR$, $1 \leq k \leq m$. The equations

$$x_1 \ldots x_n = a = y_1 \ldots y_m$$

lead us to the equations

$$P_1 \ldots P_n = aR = Q_1 \ldots Q_m$$

Corollary 2.12 implies that P_j, $1 \leq j \leq n$, Q_k, $1 \leq k \leq m$, are the non – zero prime ideals of R. By Theorem 1.4, $n = m$, and after some suitable renumbering $P_j = Q_j$, $1 \leq j \leq n$. This means that the elements x_j and y_j are associated.

We can make the above presentation unique in the following way. In every class of associated elements which consists from the prime elements we choose one fixed element. Denote the set of all selected elements by $\sigma(R)$. Then every prime element is associated with some elements of $\sigma(R)$. Theorem 2.14 implies that every non – zero element a of a principal ideal domain R can be presented in the form

$$a = ux_1 \ldots x_n,\ x_1, \ldots, x_n \in \sigma(R),\ u \in U(R)$$

(if $n = 0$, then $a = u \in U(R)$, that is, every non – zero element of R has a such form). Such a presentation is unique. Moreover, we can obtain also the following presentation

$$a = ux_1^{k_1} \ldots x_m^{k_m}, \text{ where } \{x_1, \ldots, x_m\} \subseteq \sigma(R),\ x_j \neq x_k \text{ whenever } j \neq k,\ u \in U(R).$$

Similar to ideals, it is convenient to use the following general form of the notation:

$$a = u \prod_{x \in \sigma(R)} x^{a(x)}, \text{ where } a(x) \geq 0.$$

Of course, using this form, we will assume, that only a finitely many of $a(x)$ are non – zero. Now from Theorem 2.14 we obtain

2.15. Corollary. *Let R be a principal ideal domain and a, b non – zero elements of R, $a = u \prod_{x \in \sigma(R)} x^{a(x)}$, $b = v \prod_{x \in \sigma(R)} x^{b(x)}$, $u, v \in U(R)$.*

(1) An element a divides b if and only if $a(x) \leq b(x)$ for each $x \in \sigma(R)$.

(2) **GCD**$(a, b) = \prod_{x \in \sigma(R)} x^{d(x)}$, *where* $d(x) = \min \{ a(x), b(x) \}$ *for each* $x \in \sigma(R)$.

(3) **LCM**$(a, b) = \prod_{x \in \sigma(R)} x^{m(x)}$, *where* $m(x) = \max \{ a(x), b(x) \}$ *for each* $x \in \sigma(R)$.

An integral domain is said to be a *unique factorization domain*, if every non – zero and non invertible element a is a product of prime elements of R. Moreover, if

$$x_1 \ldots x_n = a = y_1 \ldots y_m$$

are two such decompositions, then $n = m$ and after some suitably renumbering x_j and y_j are associated, $1 \leq j \leq n$.

Theorem 2.14 shows that every principal ideal domain is a unique factorization domain. But conversely is not true. Indeed, if R is an integral domain, then a polynomial ring $R[X_1, \ldots, X_n]$ is a unique factorization domain. However, it is not a principal ideal if R is not a field domain or if R is a field and $n \geq 2$.

In the conclusion of this chapter, we consider some important examples of principal ideal domains. If F is a field, then a polynomial ring $F[X]$ is a principal ideal domain. Moreover, $F[X]$ is an Euclidean domain. This example can be extended in the following way. Let F be a field, $< x >$ be an infinite cyclic group. Consider the group ring $F< x >$. Every element $g(x)$ of this rind can be written in the following form

$$g(x) = a_{-m} x^{-m} + \ldots + a_{-1} x^{-1} + a_0 + a_1 x + \ldots + a_n x^n, \text{ where } a_j \in F, -m \leq j \leq n.$$

There is the least positive integer m such that $g(x) = x^{-m} g_0(x)$ where $g_0(x)$ is a polynomial. Put $\deg (g(x)) = \deg (g_0(x))$. The ring $F< x >$ is an Euclidean domain. Indeed, let $g(x), h(x) \in F< x >$ and $h(x) \neq 0$. By above, we have $g(x) = x^{-m} g_0(x)$, $h(x) = x^{-k} h_0(x)$. Since $g_0(x)$ and $h_0(x)$ are polynomials, there are polynomials $q(x)$ and $r(x)$ such that $g_0(x) = q_0(x) h_0(x) + r_0(x)$ and either $r_0(x) = 0$ or $\deg (r_0(x)) < \deg (h_0(x))$. Then

$$g(x) = x^{-m} g_0(x) = x^{-m} q_0(x) h_0(x) + x^{-m} r_0(x).$$

If $m \geq k$, then put $x^{-m+k} q_0(x) = q(x)$, $x^{-m} r_0(x) = r(x)$. If $m < k$, then $k = m + s$ and we put $x^s q_0(x) = q(x)$, $x^{-m} r_0(x) = r(x)$. In every case,

$$g(x) = q(x) h(x) + r(x) \text{ where either } r(x) = 0 \text{ or } \deg (r(x)) < \deg (h(x)).$$

Clearly, $\deg(g(x)h(x)) = \deg(g(x)) + \deg(h(x))$. This shows that the group ring $F< x >$ is an Euclidean domain, and therefore it is a principal ideal domain.

Consider another important example.

Let D be a Dedekind domain, $P \in Spec(D)$, $\phi_n : D/P^n \longrightarrow D/P^{n+1}$ be a canonical epimorphism. Put

$$D(P^\infty) = \varinjlim proj \{ D/P^n, \phi_n \mid n \in \mathbf{N} \}.$$

Let $(\lambda_n + P^n)_{n \in N} \in U(D(P^\infty))$, then there is an element $(\mu_n + P^n)_{n \in N} \in D(P^\infty)$ such that

$$(\lambda_n + P^n)_{n \in N} (\mu_n + P^n)_{n \in N} = (1 + P^n)_{n \in N}.$$

In particular, $\lambda_n \mu_n + P^n = 1 + P^n$, so that $\lambda_1 \notin P$.

Conversely, let $(\nu_n + P^n)_{n \in N} \in D(P^\infty)$ and $\nu_1 \notin P$. Since P is a maximal ideal of D, $P + \nu_1 D = D$. We have $\nu_n - P^m = \nu_m + P^m$ for $m \leq n$; in particular, $\nu_n \notin P$ for any $n \in \mathbf{N}$. Again, $P + \nu_n D = D$. Lemma 1.18 yields that $\nu_n D + P^n = D$. Hence there are the elements $\tau_n \in D$, $\xi_n \in P^n$ such that $\nu_n \tau_n + \xi_n = 1$, or

$$1 + P^n = \nu_n \tau_n + \xi_n + P^n = \nu_n \tau_n + P^n = (\nu_n + P^n)(\tau_n + P^n).$$

Since the inverse element is unique, $\tau_n + P^m = \tau_m + P^m$ for $m \leq n$, so $(\tau_n + P^n)_{n \in N} \in D(P^\infty)$. Hence

$$U(D(P^\infty)) = \{ (\nu_n + P^n)_{n \in \mathbf{N}} \in D(P^\infty) \mid \nu_1 \notin P \}.$$

Let $(\beta_n + P^n)_{n \in N}$ be a non – zero element of $D(P^\infty)$, t a number with property $\beta_1, \beta_2, \ldots, \beta_t \in P$, but $\beta_{t+1} \notin P$. Note that $D(P^\infty) P^t = D(P^\infty) y^t$. Indeed, let $(\sigma_n + P^n)_{n \in N} \in D(P^\infty)$, $x \in P^t$. By Proposition 1.24, $P^t = Dy^t + P^{t+m}$, $m \in \mathbf{N}$, so we obtain that $x = y^t z_{t\,m} + w_{t\,m}$ where $z_{t\,m} \in D$, $w_{t\,m} \in P^{t+m}$. Then

$$((\sigma_n + P^n)_{n \in N})x = ((\sigma_n + P^n)x)_{n \in N} = (\sigma_n x + P^n)_{n \in N} =$$
$$(\sigma_n (y^t z_{t\,m} + w_{t\,m}) + P^n)_{n \in N} =$$
$$(0, \ldots, 0, \sigma_{t+1}(y^t z_{t\,1} + w_{t\,1}) + P^{t+1}), \ldots, \sigma_{t+m}(y^t z_{t\,m} + w_{t\,m}) + P^{t+m}, \ldots) =$$
$$(0, \ldots, 0, \sigma_{t+1} y^t z_{t\,1} + P^{t+1}), \ldots, \sigma_{t+m} y^t z_{t\,m} + P^{t+m}, \ldots)$$

because $w_{t\,m} \in P^{t+m}$, $m \in \mathbf{N}$.

From the equation $D(P^\infty)P^t = D(P^\infty)y^t$ we obtain that

$$(\beta_n + P^n)_{n \in \mathbf{N}} = ((\gamma_n + P^n)_{n \in N}) y^t$$

where $(\gamma_n + P^n)_{n \in \mathbf{N}} \in U(D(P^\infty))$. A D – module $D(P^\infty)$ is torsion – free (see, for example, [ND1968, 9.10, Proposition 14]), therefore the presentation

$$(\beta_n + P^n)_{n \in N} = ((\gamma_n + P^n)_{n \in N})\, y^j$$

where $(\gamma_n + P^n)_{n \in N} \in U(D(P^\infty))$, is unique. If $(\delta_n + P^n)_{n \in N}$ is another non – zero element of $D(P^\infty)$, then again

$$(\delta_n + P^n)_{n \in N} = ((\eta_n + P^n)_{n \in N})\, y^m$$

where $(\eta_n + P^n)_{n \in N} \in U(D(P^\infty))$.

Thus

$$(\beta_n + P^n)_{n \in N}\, (\delta_n + P^n)_{n \in N} = ((\gamma_n + P^n)_{n \in N})\, (\eta_n + P^n)_{n \in N}\, y^{j+m} \neq 0.$$

So $D(P^\infty)$ is an integral domain.

Let I be a non – zero ideal of $D(P^\infty)$,

$$0 \neq (\beta_n + P^n)_{n \in N} \in I,\ (\beta_n + P^n)_{n \in N} = ((\gamma_n + P^n)_{n \in N})\, y^t$$

where $(\gamma_n + P^n)_{n \in \square} \in U(D(P^\infty))$. It follows that $D(P^\infty)\, y^t \leq I$. However

$$D(P^\infty)y^t = D(P^\infty)P^t = \{ mproj\{\ D/P^{t+m},\ \mid m \in \mathbb{N}\ \}$$

and $D(P^\infty)/D(P^\infty)P^t \cong D/P^t$. By Proposition 1.24, the set of all ideal of D/P^t is

$$\{\ D/P^t,\ P/P^t,\ \ldots,\ P^{t-1}/P^t,\ <0>\ \}$$

Hence, $I/D(P^\infty)P^t$ is isomorphic to P^{t-k}/P^t for some k. In other words, $I = D(P^\infty)P^{t-k} = D(P^\infty)y^{t-k}$. This means that $D(P^\infty)$ is a principal ideal domain and the set of all non – zero ideals of $D(P^\infty)$ equals to $\{\ D(P^\infty)P^n \mid n \in \mathbb{N}\}$.

In particular, if $D = \mathbb{Z}$ and p is a prime, then $P = p\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. In this case, we come to the ring $\mathbb{Z}(p^\infty)$ of integer p – adic numbers.

Using the similar arguments we can show that the ring $F[[X]]$ of formal power series is a principal ideal domain whenever F is a field.

Chapter



Primary Decompositions of Modules over Dedekind Domains

We begin our observation of modules over Dedekind domains with very useful canonical decompositions on primary components. These decompositions generalize standard decompositions of abelina groups.

Let R be a ring, A a module over R. Put

$$t_R(A) = \{ a \in A \mid Ann_R(a) \neq < 0 > \}.$$

3.1. Lemma. *Let R be an integral domain, A an R – module. Then $t_R(A)$ is an R – submodule.*

Proof. Let $a_1, a_2 \in t_R(A)$, $L_1 = Ann_R(a_1)$, $L_2 = Ann_R(a_2)$. Then $L_1 \cap L_2 \leq Ann_R(a_1 - a_2)$. Since R does not contains zero - divisors, $L_1 L_2 \neq <0>$. From $L_1 L_2 \leq L_1 \cap L_2$ we obtain that $Ann_R(a_1 - a_2) \neq < 0 >$.

Let $a \in t_R(A)$, $0 \neq x \in R$, $u \in Ann_R(a)$. Then

$$(ax)u = a(xu) = a(ux) = (au)x = 0x = 0.$$

It follows that $Ann_R(a) \leq Ann_R(ax)$, that is $ax \in t_R(A)$.

Let R be an integral domain. A submodule $t_R(A)$ is called *the R – periodic part* of A.

An R – module A is called *R – periodic*, if $A = t_R(A)$.

An R – module A is called *R – torsion – free*, if $t_R(A) = < 0 >$.

It is easy to see that the factor – module $A/t_R(A)$ is R – torsion – free.

Let R be a ring and let A be an R – module. If S is a subset of R, then we define *the annihilator of S in A* by the rule

$$Ann_A(S) = \{ a \in A \mid ax = 0 \text{ for all } x \in S \}.$$

It is not hard to see that if the ring R is commutative. Hence the annihilator of every subset S of R is an R – submodule of A.

3.2. Lemma. *Let R be a commutative ring, A an R – module,*

$$\mathfrak{M} = \{ H \mid H \text{ is an ideal of } R \text{ such that } Ann_A(H) \neq < 0 > \}.$$

If $\mathfrak{M}$ has a maximal element $P \neq R$, then P is a prime ideal.

Proof. Let $B = Ann_A(P)$. Since R is a commutative ring, B is an R – submodule. Let x, y be elements of R such that $xy \in P$, and let $0 \neq b \in B$. We have $b(xy) = 0$. If $bx = 0$, then $xR \in \mathfrak{M}$. In this case, $xR + P \leq Ann_R(b)$; in particular, $Ann_A(xR + P) \neq < 0 >$, that is $xR + P \in \mathfrak{M}$. By the selection of P, it follows that $xR + P = P$, that is $x \in P$. If $bx \neq 0$, then $Ann_A(yR) \neq < 0 >$. Repeating the previous arguments, we obtain the inclusion $y \in P$. This means that P is a prime ideal.

3.3. Corollary. *Let R be a noetherian commutative ring, A an R – module. If $t_R(A) \neq <0>$, then R includes a non – zero prime ideal P such that $Ann_A(P) \neq < 0 >$.*

Let R be a commutative ring, A an R – module. We define the *R – assassinator of A* as the set

$$Ass_R(A) = \{ P \mid P \text{ is a prime ideal of } R \text{ such that } Ann_A(P) \neq < 0 > \}.$$

Let now D be a Dedekind domain, I be an ideal of D. Put

$$A_I = \{ a \in A \mid aI^n = <0> \text{ for some } n \in \mathbb{N} \}.$$

Clearly, A_I is a D – submodule of A.

The D – submodule A_I is called *the I – component* of A. If $A = A_I$, then A is called *an I – module*. If P is a prime ideal, then instead of an I – submodule we will say occasionally say a primary submodule.

Put

$$\Omega_{I,k}(A) = \{ a \in A \mid aI^k = <0> \}.$$

It is easy to see that $\Omega_{I,k}(A)$ is a D – submodule and

$$\Omega_{I,1}(A) \leq \Omega_{I,2}(A) \leq \ldots \leq \Omega_{I,k}(A) \leq \ldots,$$
$$A_I = \bigcup_{k \in N} \Omega_{I,k}(A).$$

We will consider some "primary" decomposition of D – periodic modules.

3.4. Lemma. *Let D be a Dedekind domain, A be a D – module, U and V be ideals of D such that $U + V = D$. Then $A_U \cap A_V = <0>$.*

Proof. Suppose that $A_U \cap A_V \neq <0>$. Then $\Omega_{U,1}(A) \cap \Omega_{V,1}(A) \neq <0>$. Choose an element $0 \neq a \in \Omega_{U,1}(A) \cap \Omega_{V,1}(A)$. The equation $U + V = R$ implies that $1 = u + v$ for some $u \in U$, $v \in V$. We have

$$a = a\,1 = a(u + v) = au + av = 0.$$

This contradiction proves the equation $A_U \cap A_V = <0>$.

3.5. Corollary. *Let D be a Dedekind domain, A be a D-module, B be a D – submodule of A, U and V be ideals of D such that $U + V = D$. If $A = A_U \oplus A_V$, then $(A/B)_U = A_U + B/B$, $(A/B)_V = A_V + B/B$, $A/B = (A/B)_U \oplus (A/B)_V$.*

Proof. It follows from the equation $A = A_U \oplus A_V$ that

$$A/B = (A_U + B)/B + (A_V + B)/B.$$

It easy to see that $(A_U + B)/B \leq (A/B)_U$, $(A_V + B)/B \leq (A/B)_V$. Lemma 3.4 implies that $(A_U + B)/B \cap (A_V + B)/B = <0>$. It follows that

$$A/B = (A_U + B)/B \oplus (A_V + B)/B$$

and

$$(A/B)_U = (A_U + B)/B,\ (A/B)_V = (A_V + B)/B.$$

3.6. Corollary. *Let D be a Dedekind domain, $\mathfrak{M}$ a family of ideals satisfying the following conditions:*

(1) if $U \in \mathfrak{M}$ then $< 0 > \neq U \neq D$;

(2) if $U, V \in \mathfrak{M}$ then $U + V = D$.

If A is a module over the ring D, then $\sum_{U \in \mathfrak{M}} A_U = \oplus_{U \in \mathfrak{M}} A_U$.

Proof. It is sufficient to prove this statement for every finite subset $\mathfrak{A} \subseteq \mathfrak{M}$. Let $|\mathfrak{A}| = k$. We shall use the induction on k. If $k = 2$, then our statement follows from Lemma 3.4. Let $k > 2$ and let $V \in \mathbf{V}$. Assume that

$$\sum_{U \in \mathfrak{A} \setminus (V)} A_U = \oplus_{U \in \mathfrak{A} \setminus (V)} A_U.$$

Suppose that $A_V \cap (\oplus_{U \in \mathfrak{A} \setminus (V)} A_U) \neq < 0 >$. Hence there are elements $a_U \in A_U$, $U \in \mathfrak{A}$ such that $a_V = \sum_{U \in \mathfrak{A} \setminus (V)} a_U$ and $a_V \neq 0$. Then $a_W \neq 0$ for some $W \neq V$. There are the numbers r_W, r_V such that $a_V V^r V = < 0 >$ and $a_W U^r W = < 0 >$. Lemma 1.18 implies that $W^r W + V^r V = D$, that is $1 = x_W + x_V$ for some elements $x_W \in W^r W$, $x_V \in V^r V$. We have

$$a_V = a_V 1 = a_V(x_W + x_V) = a_V x_W + a_V x_V = a_V x_W = \sum_{U \in \mathfrak{A} \setminus (V)} a_U x_W.$$

Since A_U is a D – submodule of A, $a_U x_W \in A_U$. Further, $a_W x_W = 0$, thus the sum $\sum_{U \in \mathfrak{A} \setminus (V)} a_U x_W$ really contains $k - 2$ summands. The induction hypothesis implies that $a_V = 0$.

3.7. Corollary. *Let D be a Dedekind domain, $\mathfrak{M}$ a family of ideals satisfying the following conditions:*

(1) if $U \in \mathfrak{M}$ then $<0> \neq U \neq D$;

(2) if $U, V \in \mathfrak{M}$ then $U + V = D$.

If A is a module over the ring D such that $A = \oplus_{U \in \mathfrak{M}} A_U$, B is a D–submodule of A, then $A/B = \oplus_{U \in \mathfrak{M}} (A_U + B/B)$.

Let D be a Dedekind domain, A a D – module, then the set $Ass_D(A)$ satisfies the conditions (1), (2) of Corollary 3.6. Consequently, we obtain the following primary decomposition.

7.8. Corollary. *Let D be a Dedekind domain, A be a D – module. If A is D – periodic, then $A = \oplus_{P \in \pi} A_P$ where $\pi = Ass_D(A)$.*

If B is a D – submodule of A, then

$$(A/B)_P = (A_P + B)/B \text{ and } A/B = \oplus_{P \in \pi} (A_P + B/B).$$

If $D = \mathbb{Z}$ and A is an abelian group, then we can consider A as a $\mathbb{Z}$ – module.

Then $Spec(\mathbb{Z}) = \{ p\mathbb{Z} \mid p$ is a prime number or $p = 0 \}$. If $P = p\mathbb{Z}$, then in this case, instead of the P – component we usually say the p – component of A. Instead of $Ass_D(A)$ we usually use the designation $\Pi(A)$. This way we come to the standard prime decomposition of an abelian group A: $A = \oplus_{p \in \Pi(A)} A_p$

Chapter



Projective Modules over Dedekind Domains

Let R be a ring. Recall that a module A over R is said to be a *free module*, if

$$A = \oplus_{\lambda \in \Lambda} A_\lambda$$

where $A_\lambda \cong_R R$ for any $\lambda \in \Lambda$.

In other words, $A_\lambda = a_\lambda R$ and $\boldsymbol{Ann}_R(a_\lambda) = < 0 >$ for any $\lambda \in \Lambda$.

It follows that every element $a \in A$ has the form $a = \sum_{\lambda \in \Lambda} a_\lambda x_\lambda$ where $x_\lambda \in R$, $\lambda \in \Lambda$, $\boldsymbol{Supp}((x_\lambda)_{\lambda \in \Lambda}) = \{ \lambda \in \Lambda \mid x_\lambda \neq 0 \}$ is finite and this presentation is unique. Here $\sum_{\lambda \in A} a_\lambda x_\lambda$ means the sum $\sum_{\lambda \in \Sigma(x)} a_\lambda x_\lambda$ where $\Sigma(x) = \boldsymbol{Supp}((x_\lambda)_{\lambda \in \Lambda})$.

Let A be an R – module. We say that the subset $B \subseteq A$ is *free*, if

$$BR = \oplus_{b \in B} bR.$$

Moreover, if $BR = A$, then B is called a *free basis* of A.

4.1. Lemma. *Let R be a ring, A an R – module.*
(1) A includes a maximal free subset.
(2) A free subset B is a maximal free subset if and only if A/BR is R – periodic.

Proof. (1) Let $\mathcal{S}$ be the family of all free subsets of A. The set $\mathcal{S}$ is ordered by inclusion. Let $\mathcal{L}$ be a linearly ordered subset of $\mathcal{S}$. It is easy to see that $\cup \mathcal{L}$ is a free subset of A. By Zorn's Lemma $\mathcal{S}$ has a maximal element.

(2) Let B be a maximal free subset of A, a $\notin$BR. In particular, $a \notin B$. Then the set $B \cup \{a\}$ is not free. This means that there are elements $b_1, \ldots, b_m \in B, x, x_1, \ldots, x_m \in R$, such that the set $\{ x, x_1, \ldots, x_m \}$ contains non – zero elements and

$$ax + b_1x_1 + \ldots + b_m x_m = 0.$$

If we assume that $x = 0$, then the set $\{ x_1, \ldots, x_m \}$ contains non – zero elements. But this means that B is not free. Hence $x \neq 0$. So $ax \in BR$ for some element $x \neq 0$ and $\boldsymbol{Ann}_R(a + BR) \neq < 0 >$.

Conversely, let $\boldsymbol{Ann}_R (a + BR) \neq < 0 >$ for every $a \notin BR$. If $0 \neq x \in \boldsymbol{Ann}_R(a + BR)$, then $ax \in BR$, that is,

$$ax = ax + b_1x_1 + \ldots + b_m x_m \text{ for suitable } b_1, \ldots, b_m \in B, x, x_1, \ldots, x_m \in R.$$

It follows that $\{ a, b_1, \ldots, b_m \}$ is not a free basis for $aR + b_1R + \ldots + b_mR$, hence $\{a\} \cup B$ is not free. This proves that B is a maximal free subset of A.

4.2. Proposition. *Let R be a ring, A be an R – module.*
(1) If A has a finite maximal free subset, then every maximal free subset of A is finite.
(2) If A does not include a finite maximal free subset then every two maximal free subset of A have the same cardinality.

Proof. (1) Let $B = \{ b_\lambda \mid \lambda \in \Lambda \}$ be a maximal free subset of A, $\{ a_1, \ldots, a_n \}$ be a finite maximal free subset of A. By Lemma 4.1, A/BR is R – periodic. Hence there are elements $0 \neq x_j \in R$, $1 \leq j \leq n$, such that $a_j x_j = \sum_{\lambda \in \Lambda} b_\lambda y_{\lambda j}$ where $\boldsymbol{Supp}\ ((y_{\lambda j})_{\lambda \in \Lambda})$ is finite, $1 \leq j \leq n$. The set

$$M = \boldsymbol{Supp}\ ((y_{\lambda 1})_{\lambda \in \Lambda}) \cup \ldots \cup \boldsymbol{Supp}\ ((y_{\lambda n})_{\lambda \in \Lambda})$$

is finite and $a_j x_j \in \oplus_{\lambda \in M} b_\lambda R$, $1 \leq j \leq n$. It follows that $a_1R + \ldots + a_nR \leq \oplus_{\lambda \in M} b_\lambda R$. Lemma 4.1 yields that $A/(a_1R + \ldots + a_nR)$ is R – periodic, hence $A/(\oplus_{\lambda \in M} b_\lambda R)$ is R – periodic. Since B is a maximal free subset of A, $M = \Lambda$, i.e. B is finite.

(2) By (1) every maximal free subset of A is infinite. Let $B = \{ b_\lambda \mid \lambda \in \Lambda \}$ and $C = \{ c_\mu \mid \mu \in M \}$ be two maximal free subsets. Then there are elements

$$0 \neq u_\lambda, x_{\lambda\mu}, 0 \neq v_\lambda, y_{\mu\lambda} \in R, \lambda \in \Lambda, \mu \in M,$$

such that

$$a_\lambda u_\lambda - \sum_{\mu \in M} b_\mu x_{\lambda\mu}, \quad b_\mu v_\mu = \sum_{\lambda \in \Lambda} a_\lambda y_{\mu\lambda}$$

where $Supp((x_{\lambda\mu})_{\mu \in M}) = M_\lambda$ and $Supp((y_{\mu\lambda})_{\lambda \in \Lambda}) = \Lambda_\mu$ are finite for every $\lambda \in \Lambda$, $\mu \in M$. By Lemma 4.1, A/BR and A/CR are R - periodic. It follows that

$$\bigcup_{\lambda \in \Lambda} M_\lambda = M \text{ and } \bigcup_{\mu \in M} \Lambda_\mu = \Lambda.$$

This means that $|\Lambda| \leq \aleph_0 |M| = |M|$ and $|M| \leq \aleph_0 |\Lambda| = |\Lambda|$ because the sets Λ and M are infinite. Thus $|\Lambda| = |M|$.

4.3. Corollary. *Let R be a ring and A be an R – module.*

(1) If A is a free R – module and A is finitely generated, then every free basis of A is finite.

(2) If A is a free R – module and A is not finitely generated, then every two free bases of A have the same cardinality.

4.4. Proposition *Let R be an integral domain, A be an R-module, $T = t_R(A)$. If $\{ a_\lambda \mid \lambda \in \Lambda \}$ is a maximal free subset of A, then $\{ a_\lambda + T \mid \lambda \in \Lambda \}$ is a maximal free subset of A/T. Conversely, if $\{ b_\mu + T \mid \mu \in M \}$ is a maximal free subset of A/T, then $\{ b_\mu \mid \mu \in M \}$ is a maximal free subset of A.*

Proof. Let $x_\lambda \in R$, $\lambda \in \Lambda$, and $Supp((x_\lambda)_{\lambda \in \Lambda})$ is finite. Suppose that $\sum_{\lambda \in \Lambda}(a_\lambda + T)x_\lambda = T$. In other words, $\sum_{\lambda \in \Lambda} a_\lambda x_\lambda \in T$. Then there exists an element $0 \neq x \in R$ such that $(\sum_{\lambda \in \Lambda} a_\lambda x_\lambda)x = 0$. From the selection of A it follows that $x_\lambda x = 0$ for every $\lambda \in \Lambda$. Since R does not contain zero – divisors, $x_\lambda = 0$ for every $\lambda \in \Lambda$. This means that $\{ a_\lambda + T \mid \lambda \in \Lambda \}$ is a free subset of A/T. Conversely, let $y_\mu \in K$, $\mu \in M$, $Supp((y_\mu)_{\mu \in M})$ be finite and $\sum_{\mu \in M} b_\mu y_\mu = 0$. Then

$$T = \sum_{\mu \in M} b_\mu y_\mu + T = \sum_{\mu \in M} (b_\mu + T) y_\mu.$$

Since $\{ b_\mu + T \mid \mu \in M \}$ is a free subset of A/T, $y_\mu = 0$ for every $\mu \in M$. This proves that $\{ b_\mu \mid \mu \in M \}$ is a free subset of A.

Now suppose that $\{a_\lambda + T \mid \lambda \in \Lambda\}$ is not a maximal free subset of A/T. Then there exists an element $c + T \neq T$ such that $\{ c + T, a_\lambda + T \mid \lambda \in \Lambda\}$ is free. But we have already proved that in this case the set $\{c, a_\lambda \mid \lambda \in \Lambda\}$ is free. This contradiction proves that $\{a_\lambda + T \mid \lambda \in \Lambda\}$ is a maximal free subset of A/T. Similarly, if $\{b_\lambda + T \mid \lambda \in \Lambda\}$ is a maximal free subset of A/T, then $\{ b_\lambda \mid \lambda \in \Lambda \}$ is a maximal free subset of A.

4.5. Lemma. *Let R be an integral domain, A be an R-module and A be R-torsion-free, F be a field of fractions of R, $E = A \otimes_R F$. The subset $\{ b_\lambda \mid \lambda \in \Lambda \}$ is a free maximal subset of A if and only if it is a basis for E.*

Proof. Let $B = \{ b_\lambda \mid \lambda \in \Lambda \}$, $e \in E$. Then $ex \in A$ for some $0 \neq x \in r$. By Lemma 4.1, A/BR is R-periodic, so $(ex)y \in BR$ for some $0 \neq y \in R$. This means that $e = e(xy)(xy)^{-1} \in BF$. This implies that $BF = E$.

Let $\alpha_\lambda \in F$, $\lambda \in \Lambda$, $Supp((\alpha_\lambda)_{\lambda \in \Lambda})$ be finite, and $\sum_{\lambda \in \Lambda} b_\lambda \alpha_\lambda = 0$. There is an element $0 \neq u \in R$ such that $\alpha_\lambda u \in R$ for all $\lambda \in \Lambda$. Then

$$0 = 0u = (\sum_{\lambda \in \Lambda} b_\lambda \alpha_\lambda)u = \sum_{\lambda \in \Lambda} b_\lambda (\alpha_\lambda u).$$

It follows that

$$\alpha_\lambda u = 0 \text{ or } \alpha_\lambda = (\alpha_\lambda u)u^{-1} = 0 \text{ for all } \lambda \in \Lambda.$$

The converse assertion is obvious.

The following corollary is an immediate consequence of Proposition 4.4 and Lemma 4.5.

4.6. Corollary. *Let R be an integral domain, A be an R-module, B and C be the maximal free subset of A. Then $|B| = |C|$.*

4.7. Corollary. *Let R be an integral domain, A be a free R-module. If B and C are two free basis of A, then $|B| = |C|$.*

Let R be an integral domain, A be an R-module, B be a maximal free subset of A. Then $|B|$ is called the *torsion-free R-rank* of A or, shortly, the *0-rank* of A. The 0-rank of A we shall denote by $r_0(A)$.

Let R be a ring, A, B be R-modules, $\alpha: A \longrightarrow B$ be an R-monomorphism. An R-epimorphism $\beta: B \longrightarrow A$ is called a *backmapping* if $\alpha\beta = \varepsilon_A$ is an identity permutation of A.

Similarly, if $\alpha: A \longrightarrow B$ is an R-epimorphism, then an R-monomorphism $\beta: B \longrightarrow A$ is called a *backmapping*, if $\beta\alpha = \varepsilon_B$.

4.8. Lemma. *Let R be a ring, A and B be R-modules.*

(1) If there exist an R-homomorphisms $\alpha: A \longrightarrow B$ and $\beta: B \longrightarrow A$ such that $\alpha\beta = \varepsilon_A$, then α is a monomorphism, β is an epimorphism and $B = Im\alpha \oplus Ker\beta$. In particular, A is isomorphic to a direct summand of B.

(2) If $B = A_1 \oplus A_2$ and $A_1 \cong A$, then there exists an R-monomorphism $\alpha: A \longrightarrow B$ and an R-epimorphism $\beta: B \longrightarrow A$ such that $\alpha\beta = \varepsilon_A$.

Proof. (1). Let $b \in Im\,\alpha \cap Ker\,\beta$. Then $b = a\alpha$ for some $a \in A$ and $b\beta = 0$, that is,

$$0 = b\beta - (a\alpha)\beta = a(\alpha\beta) = a\varepsilon_A = a.$$

It follows that $b = 0\alpha = 0$, and $\mathit{Im}\ \alpha \cap \mathit{Ker}\ \beta = < 0 >$. Let $\beta \in B$. We have

$$(b - (b\beta)\alpha)\beta = b\beta - ((b\beta)\alpha)\beta = b\beta - (b\beta)(\alpha\beta) = b\beta - (b\beta)\varepsilon_A = 0.$$

Therefore $b - (b\beta)\alpha \in \mathit{Ker}\ \beta$. Since $(b\beta)\alpha \in \mathit{Im}\ \alpha$, $b \in \mathit{Im}\ \alpha + \mathit{Ker}\ \beta$, that is, $B = \mathit{Im}\ \alpha + \mathit{Ker}\ \beta$, and hence

$$B = \mathit{Im}\ \alpha \oplus \mathit{Ker}\ \beta.$$

Furthermore, if a is an arbitrary element of A, then $a = a\varepsilon_A = a(\alpha\beta) = (a\alpha)\beta$. This means that β is an epimorphism. Finally, if $a \in \mathit{Ker}\ \alpha$, then we obtain

$$a = a\varepsilon_A = a(\alpha\beta) = (a\alpha)\beta = 0\beta = 0,$$

so that α is a monomorphism.

(2) Let $\phi: A \longrightarrow A_1$ be an R – isomorphism, $\pi_1: B \longrightarrow A_1$, $\pi_2: B \longrightarrow A_2$ be projection mappings. Put $\alpha = \phi\varepsilon$ where $\varepsilon: A_1 \longrightarrow B$ is an identity imbedding, $\beta = \pi_1 \cdot \phi^{-1}$. If $a \in A$, then

$$a(\alpha\beta) = (a\alpha)\beta = (a\phi\varepsilon)(\pi_1\phi^{-1}) = ((a\phi\varepsilon)\pi_1)\ \phi^{-1} = (a\phi)\phi^{-1} = a.$$

A module P over a ring R is called *projective*, if for every R – epimorphism $\beta: A \longrightarrow B$ and an R – homomorphism $\alpha: P \longrightarrow B$ there exists an R – homomorphism $\gamma: P \longrightarrow A$ such that $\alpha = \beta\gamma$.

We will consider now the relationships between projective and free R – modules.

4.9. Proposition. *Let R be a ring, C be a free R – module with a free basis $\{ c_\lambda \mid \lambda \in \Lambda \}$.*

(1) If A is an R – module, $\{ a_\lambda \mid \lambda \in \Lambda \}$ is a subset of A, then there exists a unique R – homomorphism $\phi: C \longrightarrow A$ such that $c_\lambda\phi = a_\lambda$, $\lambda \in \Lambda$.

(2) Every R – module is a homomorphic image of some free R – module.

(3) Every finitely generated R – module is a homomorphic image of a free module having finite free basis.

Proof. Put $(\sum_{\lambda \in \Lambda} c_\lambda x_\lambda)\phi = \sum_{\lambda \in \Lambda} a_\lambda x_\lambda$. Since a presentation of every element of C is unique, ϕ is a mapping. It is easy to see that ϕ is an R – homomorphism.

(2) and (3) follow from (1).

4.10. Proposition. *Let R be a ring, C be a free R – module. Then C is a projective module.*

Proof. Let A, B be R – modules, $\beta: A \longrightarrow B$ be an R – epimorphism, $\alpha: C \longrightarrow B$ be an R – homomorphism, $\{c_\lambda \mid \lambda \in \Lambda\}$ a free basis of C. There exists an

element $a_\lambda \in A$ such that $a_\lambda\beta = c_\lambda\alpha$ for each $\lambda \in \Lambda$. By Proposition 4.9, there exists an R – homomorphism $\gamma: C \longrightarrow A$ such that $c_\lambda\gamma = a_\lambda$ $\lambda \in \Lambda$. Now

$$c_\lambda(\gamma\beta) = (c_\lambda\gamma)\beta = a_\lambda\beta = c_\lambda\alpha \text{ for all } \lambda\in\Lambda.$$

Since $C = \sum_{\lambda \in \Lambda} c_\lambda R$, we obtain the equation $\gamma\beta = \alpha$.

Let A, B, C be R – modules. The sequence of R – homomorphisms

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is called a *short exact sequence*, if $C = Im\ \beta \cong B/Ker\ \beta$ and $Ker\ \beta = Im\ \alpha \cong A$.

In particular, if A is an R – submodule of B, $\nu: B \longrightarrow B/A$ is a natural homomorphism, then $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ is a short exact sequence.

4.11. Lemma. *Let R be a ring, A, B, C be R – modules,*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence. The epimorphism β has a back monomorphism $\gamma: C \longrightarrow B$ if and only if the monomorphism α has a back epimorphism $\delta: B \longrightarrow A$.

Proof. Suppose that β has a back mapping $\gamma: C \longrightarrow B$, that is, $\gamma\beta = \varepsilon_C$. By Lemma 4.8, $B = Ker\ \beta \oplus Im\ \gamma$ and $Im\ \gamma \cong C$. Since $Ker\ \beta = Im\ \alpha \cong A$, $B \cong A \oplus C$. Let π be a projection mapping of B on $Ker\ \beta = Im\ \alpha$. Define $\nu: Im\ \alpha \longrightarrow A$ by the rule $(a\alpha)\nu = a$ for all $a \in A$. Put $\delta = \pi\nu$. Then δ is a back epimorphism for α.

Conversely, suppose that α has a back epimorphism $\delta: B \longrightarrow A$, that is $\alpha\delta = \varepsilon_A$. By Lemma 4.8, $B = Im\ \alpha \oplus Ker\ \delta$. Moreover, $Ker\ \delta \cong B/Im\ \alpha = B/Ker\ \beta \cong C$, and therefore, $B = Im\ \alpha \oplus Ker\ \delta \cong A \oplus C$. If $b \in B$, then $b = e + d$ where $e \in Ker\ \beta$, $d \in Ker\ \delta$. Define the mapping $\beta_0: Ker\ \delta \longrightarrow C$ by the rule $d\beta_0 = d\beta$, $d \in Ker\ \delta$. Clearly, $Ker\ \beta_0 = <0>$ and $Im\ \beta_0 = Im\ \beta = C$, i.e. β_0 is an isomorphism of $Ker\ \delta$ on C. Put $\gamma = \beta_0^{-1}$. If $c \in C$, then $c(\gamma\beta) = (c\gamma)\beta = (c\beta_0^{-1})\beta = c$.

A short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is called a *split short exact sequence*, if the epimorphism β has a back monomorphism (or, equivalently, by Lemma 4.11, the monomorphism α has a back epimorphism).

Now we can procide with the following characterization of projective modules.

4.12. Theorem. *Let R be a ring, A, B, P be R – modules. The following conditions are equivalent:*

(1) P is a projective R – module;

(2) every short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ splits;

(3) P is isomorphic to a direct summand of some free R – module.

Proof. (1) $\Rightarrow$ (2). Since P is a projective R – module, then there exists a homomorphism γ: $P \longrightarrow B$ such that $\gamma\beta = \varepsilon_P$. This means that the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ is a split short exact sequence.

(2) $\Rightarrow$ (3). By Proposition 4.9, there exist a free module C and an epimorphism ν: $C \longrightarrow P$. We consider the short exact sequence $0 \longrightarrow A \longrightarrow C \longrightarrow P \longrightarrow 0$. Since it splits, Lemma 4.8 implies that $C = Im\ \tau \oplus Ker\ \nu$, where τ is a back monomorphism for ν. Then $Im\ \tau \cong C/Ker\ \nu \cong P$.

(3) $\Rightarrow$ (1). Let C be a free R – module, $C = C_1 \oplus C_2$ and $C_1 \cong_R P$, β_1: $A \longrightarrow B$ be an R – epimorphism, α_1: $P \longrightarrow B$ be an R – homomorphism. Using Lemma 4.8 we have an R – monomorphism σ: $P \longrightarrow C$ and an R – epimorphism τ: $C \longrightarrow P$ such that $\sigma\tau = \varepsilon_P$. Put $\alpha_2 = \tau\alpha_1$. Proposition 4.10 shows that C is a projective R – module, and hence there exists a homomorphism γ_2: $C \longrightarrow A$ such that $\gamma_2\beta_1 = \alpha_2$. Put $\gamma_1 = \sigma\gamma_2$. We have now

$$\gamma_1\beta_1 = (\sigma\gamma_2)\beta_1 = \sigma(\gamma_2\beta_1) = \sigma\alpha_2 = \sigma(\tau\alpha_1) = (\sigma\tau)\alpha_1 = \varepsilon_P\alpha_1 = \alpha_1.$$

4.13. Lemma. *Let R be a ring.*

(1) Every R – module is a homomorphic image of some projective R – module.

(2) Every finitely generated R – module is a homomorphic image of some finitely generated projective R – module.

(3) If P_λ is a projective R – module for any $\lambda \in \Lambda$, then $\oplus_{\lambda \in \Lambda} P_\lambda$ is a projective R – module.

Proof. (1) and (2) follow from Proposition 4.9 and Proposition 4.10. Theorem 4.12 implies that $P_\lambda \cong_R D_\lambda$ where D_λ is a direct summand of some free R – module C_λ, $C_\lambda = D_\lambda \oplus E_\lambda$. Put $C = \oplus_{\lambda \in \Lambda} C_\lambda$. Then $C = (\oplus_{\lambda \in \Lambda} D_\lambda) \oplus (\oplus_{\lambda \in \Lambda} E_\lambda)$. Obviously, C is a free R – module. Since $\oplus_{\lambda \in \Lambda} P_\lambda \cong_R \oplus_{\lambda \in \Lambda} D_\lambda$, Theorem 4.12 claims that $\oplus_{\lambda \in \Lambda} P_\lambda$ is a projective R – module.

4.14. Lemma. *Let R be an integral domain, F be the field of fractions for R, L be an ideal of R. Then there exists a $\mathbb{Z}$ – isomorphism θ: $L^{-1} \longrightarrow Hom_R(L, R)$ such that $a(x\theta) = xa$ for every $x \in L^{-1}$, $a \in L$.*

Proof. Let $x \in L^{-1}$, and define the mapping ϕ_x: $L \longrightarrow R$ by the rule: $a\phi_x = xa$ for every $a \in L$. Obviously

$$(a + b)\phi_x = x(a + b) = xa + xb = a\phi_x + b\phi_x,\ (ca)\phi_x = x(ca) = c(xa) = c(a\phi_x).$$

Thus ϕ_x is an R – homomorphism. We define θ by the rule $x\theta = \phi_x$ for every $x \in L^{-1}$. Since

$$a\varphi_{x+y} = (x + y)a = xa + ya = a\phi_x + a\phi_y = a(\phi_x + \phi_y),\ (x + y)\theta = x\theta + y\theta,$$

so θ is a $\mathbb{Z}$ – homomorphism. Let $x \in Ker\ \theta$, that is $a\phi_x = 0$ for each $a \in A$. Since R does not contain zero – divisors, $Lx = <0>$ implies $x = 0$. So θ is a $\mathbb{Z}$ – monomorphism.

Finally, let $\chi \in Hom_R(L, R)$, $a, b \in L \setminus < 0 >$. Then

$$(a\chi)b = (ab)\chi = (ba)\chi = (b\chi)a,$$

so that $(a\chi)a^{-1} = (b\chi)b^{-1}$. This equation impliess that $(a\chi)a^{-1}$ is the same element for every $0 \neq a \in L$. Put $(a\chi)a^{-1} = x$. We have $(a\chi)a^{-1} = x$, that is $a\chi = xa$ for every $0 \neq a \in L$.

Since $0\chi = 0 = x \cdot 0$, this means that $\chi = \phi_x$. Furthermore, $a\chi \in R$, i.e. $ax \in R$ for every $a \in A$. Therefore $x \in L^{-1}$, and this implies that $\chi = (\phi_x)\theta$ and θ is a $\mathbb{Z}$ – isomorphism.

A module A over a ring R is said to be *hereditary* if A and all its submodules are projective.

Notice that submodules of hereditary modules are necessarily hereditary.

A ring R is said to be *hereditary* if all its right ideals are projective modules. Obviously, this occurs if and only if the R – module R is hereditary.

4.15. Proposition. *Let R be an integral domain, L be a non – zero ideal of R. The R – module L is projective if and only if L is an invertible ideal. In particular, R is a heredity ring if and only if all its non – zero ideals are invertible.*

Proof. Suppose at first that L is invertible. Then there are elements $a_1, \ldots, a_m \in L$, $x_1, \ldots, x_m \in L^{-1}$ such that $1 = a_1x_1 + \ldots + a_mx_m$. Let C be a free R – module with a free basis $\{c_1, \ldots, c_m\}$. By Proposition 4.9, there exists an R – homomorphism $\pi: C \longrightarrow L$ such that $c_j\pi = a_j$, $1 \leq j \leq m$. Define the mapping $\alpha: L \longrightarrow C$ by the rule

$$a\alpha = c_1(ax_1) + \ldots + c_m(ax_m),\ a \in L.$$

The mapping α is an R – homomorphism because R is commutative. If $a \in L$, then

$$a(\alpha\pi) = (a\alpha)\pi = (c_1ax_1 + \ldots + c_max_m)\pi = (c_1\pi)ax_1 + \ldots + (c_m\pi)ax_m = a_1ax_1 + \ldots + a_max_m = a(a_1x_1 + \ldots + a_mx_m) = a \cdot 1 = a.$$

So $\alpha\pi = \varepsilon_L$. Lemma 4.8 implies that $C = Im\ \alpha \oplus Ker\ \pi$ and $L \cong Im\ \alpha$. Theorem 4.12 implies that L is a projective module.

Conversely, let L be a projective module. Theorem 4.12 implies that there is a free R – module C_L such that $C_L = L \oplus A$ for some R – submodule A. Let $\{c_\lambda / \lambda \in \Lambda\}$ be a free basis of C_L. If $a \in L$, then $a = \sum_{\lambda \in \Lambda} c_\lambda\tau_\lambda(a)$ for some elements $\tau_\lambda(a) \in R$, $\lambda \in \Lambda$. If b is another element of L, then

$$a + b = \sum_{\lambda \in \Lambda} c_\lambda\tau_\lambda(a) + \sum_{\lambda \in \Lambda} c_\lambda\tau_\lambda(b) = \sum_{\lambda \in \Lambda} c_\lambda(\tau_\lambda(a) + \tau_\lambda(b)).$$

Since $a + b = \sum_{\lambda \in \Lambda} c_\lambda\tau_\lambda(a + b)$ and $\{c_\lambda / \lambda \in \Lambda\}$ is a free basis of C_L, we obtain the equation $\tau_\lambda(a + b) = \tau_\lambda(a) + \tau_\lambda(b)$, $\lambda \in \Lambda$. Similarly, $\tau_\lambda(da) = d\tau_\lambda(a)$ for each $d \in R$, $\lambda \in \Lambda$. It follows that the mapping $a \longrightarrow \tau_\lambda(a)$, $\lambda \in \Lambda$, is an R – homomorphism. Lemma 4.14 implies that there are elements $y_\lambda \in L^{-1}$ such that $\tau_\lambda(a) = y_\lambda a$ for every

$a \in L$, $\lambda \in \Lambda$. In other words $a = \sum_{\lambda \in \Lambda} c_\lambda(y_\lambda a)$. Let $0 \neq b \in I$, $\Lambda_b = \mathbf{Supp}((\tau_\lambda(b))_{\lambda \in \Lambda})$. Then Λ_b is finite, say $\Lambda_b = \{ \lambda_1, \ldots, \lambda_k \}$. In particular, $\tau_\lambda(b) = 0$ for $\lambda \in \Lambda \setminus \Lambda_b$. This means that $y_\lambda b = 0$ for $\lambda \in \Lambda \setminus \Lambda_b$, that is $y_\lambda = 0$ for $\lambda \in \Lambda \setminus \Lambda_b$. So $b = \sum_{\lambda \in \Lambda} c_\lambda(y_\lambda b)$. Let π be a projection mapping of C_I on L. Since π is an R – homomorphism,

$$b = b\pi = (\sum_{\lambda \in \Lambda} c_\lambda(y_\lambda b))\pi = \sum_{\lambda \in \Lambda} b_\lambda(y_\lambda b).$$

Multiplying both sides of this equation by y^{-1} (R is a subring of its fields of fractions), we obtain then $1 = \sum_{\lambda \in \Lambda} b_\lambda y_\lambda$. It follows that $R = L L^{-1}$.

4.16. Theorem. *Let R be an integral domain. Then R is a hereditary ring if and only if R is a Dedekind domain.*

Proof. Suppose that R is a hereditary ring. Proposition 4.15 implies that every non – zero ideal of R is invertible. It follows that every fractional ideal of R is invertible, thus R is a Dedekind domain. The converse assertion is obvious.

Now we can describe the projective modules over Dedekind domains.

4.17. Lemma. *Let R be a ring, A be an R – module, B be a submodule of A, and let A has an ascending series of submodules*

$$B = A_0 \leq A_1 \leq \ldots \leq A_\alpha \leq A_{\alpha+1} \leq \ldots A_\gamma = A.$$

If each factor $A_{\alpha+1}/A_\alpha$ of this series is a projective module for $\alpha < \gamma$, then $A = B \oplus C$ for some R – submodule C.

Proof. We shall show that for every $\alpha \leq \beta$ there exists an R – submodule C_α such that $B_\alpha = B \oplus C_\alpha$ and $C_\alpha \leq C_{\alpha+1}$.

Consider the short exact sequence

$$0 \longrightarrow B \longrightarrow A_1 \longrightarrow A_1/B \longrightarrow 0.$$

Since A_1/B is a projective R– module, Theorem 4.12 implies that this sequence is splitting. So $A_1 = B \oplus C_1$ for some R– submodule $C_1 \cong A_1/A_0$. For $\alpha = 1$ all is proved.

Let $\alpha > 1$ and we have already constructed R – submodules C_β for $\beta < \alpha$. If $\alpha - 1$ exists, then consider the short exact sequence

$$0 \longrightarrow A_{\alpha-1} \longrightarrow A_\alpha \longrightarrow A_\alpha/A_{\alpha-1} \longrightarrow 0.$$

Since $A_\alpha/A_{\alpha-1}$ is a projective R– module, Theorem 4.12 implies that $A_\alpha = A_{\alpha-1} \oplus D_\alpha$ for some R– submodule D_α. By the induction hypothesis we have $A_{\alpha-1} = B \oplus C_{\alpha-1}$, so $A_\alpha = B \oplus C_{\alpha-1} \oplus D_\alpha$. Put $C_\alpha = C_{\alpha-1} \oplus D_\alpha$.

Let α be a limit ordinal. Put $C_\alpha = \cup_{\beta<\alpha} C_\beta$. Then we have

$$B \cap C_\alpha = B \cap (\cup_{\beta<\alpha} A_\beta) = \cup_{\beta<\alpha}(B \cap C_\beta) = <0>.$$

Let $a \in A_\alpha$. Then $a \in A_\beta$ for some $\beta < \alpha$, so that $a \in B \oplus C_\beta \leq B \oplus C_\alpha$. It follows that $A_\alpha = B \oplus C_\alpha$. For $\alpha = \gamma$ we obtain $A = A_\gamma = B \oplus C_\gamma = B \oplus C$.

4.18. Corollary. *Let R be a ring, A be an R-module having an ascending series of submodules, the factors of which are projective R – modules. Then $A = \oplus_{\lambda \in \Lambda} C_\lambda$ where C_λ is a projective R – module, $\lambda \in \Lambda$. In particular, A is a projective R – module.*

4.19. Corollary *Let R be a ring, $\{ A_\lambda \mid \lambda \in \Lambda \}$ be a family of R – modules, $A = \oplus_{\lambda \in \Lambda} A_\lambda$.*

(1) If A_λ is a hereditary R – module for every $\lambda \in \Lambda$, then A is a hereditary R – module.

(2) If B is an R – submodule of A, then $B = \oplus_{\lambda \in \Lambda} B_\lambda$ where B_λ is isomorphic to some R – submodule of A_λ, $\lambda \in \Lambda$.

Proof. We can assume that the set Λ is well – ordered, $\Lambda = \{ \alpha \mid \alpha < \gamma \}$. Let $\pi_\alpha: A \longrightarrow A_\alpha$ be a projection mapping of A on A_α. For every $0 \neq a \in A$ we define $deg(a)$ as $max\{ \alpha \mid \pi_\alpha(a) \neq 0 \}$. The set $\{ \pi_\alpha(a) \mid \alpha < \gamma \}$ contains only finitely many non – zero elements, so $deg(a)$ is defined for every non – zero element a. For $a = 0$ we put $deg(a) = -\infty$.

Let B be an R – submodule of A. For every $\alpha < \gamma$ put

$$E_\alpha = \{ b \in B \mid deg(b) \leq \alpha \}.$$

Obviously, $E_\alpha \leq E_{\alpha+1}$ and if α is a limit ordinal, then $E_\alpha = \cup_{\beta < \alpha} E_\beta$, in particular, $B = \cup_{\beta < \alpha} E_\alpha$. Put $B = E_\gamma$. Moreover, $E_\alpha \leq \oplus_{\beta \leq \alpha} A_\beta = D_\alpha$, $\alpha < \gamma$. Therefore

$$E_{\alpha+1}/E_\alpha = E_{\alpha+1}/(E_\alpha \cap D_\alpha) \cong (E_{\alpha+1} + D_\alpha)/D_\alpha \leq D_{\alpha+1}/D_\alpha \cong A_{\alpha+1}.$$

Since $A_{\alpha+1}$ is a hereditary module, then every submodule of $A_{\alpha+1}$ is projective. Corollary 4.18 implies that $B = \oplus_{\alpha \leq \gamma} B_\alpha$, where B_α is isomorphic to some submodule of A_α, in particular, B_α is a projective R – module, $\alpha \leq \gamma$. So, by Lemma 4.13, B is a projective R– module. This means that A is a hereditary R – module.

4.20. Theorem. *If R is a hereditary ring, then every free R – module is hereditary. Every submodule of the free R – module (in particular, every projective module) is a direct sum of submodules, each of which is isomorphic to some right ideal of R.*

Proof. Let C be a free R – module. Then $C = \oplus_{\lambda \in \Lambda} C_\lambda$ where $C_\lambda \cong R$ for any $\lambda \in \Lambda$. Corollary 4.19 implies that C is a hereditary R – module. If B is an R – submodule of C, then by Corollary 4.19, we obtain that $B = \oplus_{\lambda \in \Lambda} B_\lambda$ where B_λ is isomorphic to some R – submodule of R, that is, to a right ideal of R, $\lambda \in \Lambda$.

4.21. Corollary. *Let D be a Dedekind domain. Then every free D– module is hereditary. Every submodule of a free module (in particular, a projective module) is a direct sum of submodules, each of which is isomorphic to some ideal of D.*

In fact, it follows from Theorem 4.16 that D is a hereditary ring.

4.22. Corollary. *Let R be a principal ideal domain, C be a free R – module. Then R is a hereditary ring and every submodule of C is free.*

Proof. The ring R is hereditary since every principal ideal domain is a Dedekind domain. Let B be a submodule of C, then $B = \oplus_{\lambda \in \Lambda} B_\lambda$ where $B_\lambda \cong_R L_\lambda$ for some ideal L_λ of the riing R, $\lambda \in \Lambda$ by Corollary 4.21. Let $L_\lambda = a_\lambda K$, then the mapping ϕ_λ: $R \longrightarrow L_\lambda$ which is defined by the rule $x\phi_\lambda = a_\lambda x$, $x \in R$, is an epimorphism. Since R does not contain zero – divisors, $Ker\ \phi_\lambda = < 0 >$, that is, $L_\lambda \cong_R R$.

4.23. Lemma. *Let R be a ring, U and V be right ideals of R.*

(1) If $U + V = R$, then R – modules $U \oplus V$, and $R \oplus (U \cap V)$ are isomorphic.

(2) If A and B are R – modules, $A \cong B$, then $AL \cong_R BL$ for every right ideal L of the ring R.

Proof. (1). Define the mapping ϕ: $U \oplus V \longrightarrow K$ by the rule $(u, v)\phi = u - v$. It is easy to see that ϕ is an R – homomorphism. The equation $U \oplus V = R$ implies that ϕ is an epimorphism. Since R is a projective R – module, $U \oplus V = R \oplus W$, where $W = Ker\ \phi$. If $(u, v) \in Ker\ \phi$, then $u = v$.

Now we define the mapping ψ: $U \cap V \longrightarrow W$ by the rule $x\psi = (x, x)$. Obviously, ψ is an R – isomorphism, so $W \cong U \cap V$.

(2) Let θ: $A \longrightarrow B$ be an R – isomorphism, then $(AL)\theta = BL$, so $AL \cong_R BL$.

4.24. Lemma. *Let D be a Dedekind domain.*

(1) If A and B are non – zero fractional ideals of D, then $A \oplus B$ and $D \oplus AB$ are isomorphic as D – modules.

(2) If $A_1, \ldots, A_k, B_1, \ldots, B_r$ are non – zero fractional ideals of D, then D – modules $A_1 \oplus \ldots \oplus A_k$ and $B_1 \oplus \ldots \oplus B_r$ are isomorphic if and only if $k = r$ and $A_1 \ldots A_k = xB_1 \ldots B_k$ for some element $0 \neq x \in F$, where F is the field of fractions of the ring D.

Proof. (1) There are elements $0 \neq a \in A$ and $0 \neq b \in B$ such that $aA \leq D$ and $bB \leq D$, moreover, aA and bB are ideals of D. The mappings $c \longrightarrow ac$, $c \in A$ and $d \longrightarrow db$, $d \in B$, are D – monomorphisms, that is $A \cong_D aA$, $B \cong_D bB$. Similarly, $abAB$ is a non – zero ideal of D and $AB \cong_D (aA)(bB) = abAB$. Therefore we can assume that A, B are integral ideals of D.

Note that if $A + B = D$, then $AB = A \cap B$ by Proposition 1.17. Lemma 4.23 implies that in this case, $A \oplus B$ and $D \oplus AB$ are isomorphic as D – modules.

By Corollary 1.13

$$A = P_1^{h_1} \ldots P_t^{h_t} \text{ where } P_i \neq P_j \text{ for } i \neq j,\ P_i \in Spec(D),\ h_i \geq 0,\ 1 \leq i \leq t.$$

We shall apply induction on the number $m(A) = h_1 + \ldots + h_t$. If $m(A) = 0$, then $A = D$ and $AB = B$, so the statement is trivial. If $m(A) = 1$, then A is a maximal ideal of D. It follows from $BB^{-1} = D$ that there is an element $0 \neq x \in B^{-1}$ such that $xB \leq A$. Now $A + xB = D$, so that $A \oplus xB \cong_D D \oplus xAB$. But $xB \cong_D B$ and $AB \cong_D xAB$.

Let $m(A) > 1$. If $A + B = D$, the result follows from **Lemma 4.23**. So we can suppose that $A + B \leq P$ for some maximal ideal P of D. We have

$$P_1^{h_1} \ldots P_t^{h_t} = A \leq A + B \leq P.$$

Proposition 1.17 and Theorem 1.14 imply that $P = P_j$ for some j, $1 \leq j \leq t$. Say $P = P_1$. Then from $A \leq P$ and $B \leq P$ we obtain that $A^{-1} \geq P_1^{-1}$ and $B^{-1} \geq P_1^{-1}$. Hence AP_1^{-1} and BP_1^{-1} are ideals of D. In particular, $m(AP_1^{-1}) = m(A) - 1$. By the induction hypothesis we have

$$AP_1^{-1} \oplus BP_1^{-1} \cong_D D \oplus (AP_1^{-1})(BP_1^{-1}).$$

Lemma 4.23 implies that $A \cong_D (AP_1^{-1})P_1$, $B \cong_D (BP_1^{-1})P_1$, and

$$A \oplus B = (AP_1^{-1})P_1 \oplus (BP_1^{-1})P_1 \cong_D$$
$$\cong_D (AP_1^{-1} \oplus BP_1^{-1})P_1 \cong_D (D \oplus (AP_1^{-1})(BP_1^{-1}))P = P_1 \oplus A(BP_1^{-1}).$$

Since P_1 is a maximal ideal, $m(P) = 1$. We have already considered this case. So

$$P_1 \oplus A(BP_1^{-1}) \cong_D D \oplus A(BP_1^{-1})P_1 = D \oplus AB.$$

Consequently, $A \oplus B \cong_D D \oplus AB$.

(2) From (1) we obtain that

$$A_1 \oplus A_2 \oplus \ldots \oplus A_k \cong_D (D \oplus A_1A_2) \oplus A_3 \oplus \ldots \oplus A_k \cong_D$$
$$\cong_D (D \oplus D \oplus A_1A_2A_3) \oplus A_4 \oplus \ldots \oplus A_k \cong_D \ldots \cong_D$$
$$\cong_D \underbrace{D \oplus \ldots \oplus D}_{k-1} \oplus A.$$

where $A = A_1A_2 \ldots A_k$. Similarly

$$B_1 \oplus B_2 \oplus \ldots \oplus B_r \cong_D \underbrace{D \oplus \ldots \oplus D}_{r-1} \oplus B$$

where $B = B_1B_2 \ldots B_r$. Therefore if $k = r$ and $A = xB$ for some $0 \neq x \in F$, then D – modules $A_1 \oplus A_2 \oplus \ldots \oplus A_k$ and $B_1 \oplus B_2 \oplus \ldots \oplus B_r$ are isomorphic.

Conversely, let $\theta: A_1 \oplus A_2 \oplus \ldots \oplus A_k \longrightarrow B_1 \oplus B_2 \oplus \ldots \oplus B_r$ be an isomorphism. If C is a non – zero D – submodule of F and $0 \neq c \in C$, then $C_1 = c^{-1}C$ is a non – zero submodule of F and $1 \in C_1$. Moreover, D – submodules C and C_1 are isomorphic. Therefore we can assume that $1 \in A_1$, $1 \in B_1$, $1 \leq i \leq k$, $1 \leq j \leq r$. For every i, $1 \leq i \leq k$, we put

$$(0, 0, \ldots, 0, 1, 0, \ldots, 0)\theta = (b_{i1}, b_{i2}, \ldots, b_{ir})$$
$$\Downarrow$$
$$i$$

$b_{ij} \in B_j$, $1 \le j \le r$. For each $a_i \in A_i$ we have

$$(0, 0, \ldots, 0, a_i, 0, \ldots, 0)\theta = ((0, 0, \ldots, 0, 1, 0, \ldots, 0)a_i)\theta =$$
$$(0, 0, \ldots, 0, 1, 0, \ldots, 0)\theta \cdot a_i = (b_{i1}, b_{i2}, \ldots, b_{ir})a_i =$$
$$(b_{i1}a_i, b_{i2}a_i, \ldots, b_{ir}a_i).$$

In particular, $a_i b_{ij} \in B_j$, $1 \le j \le r$. It follows that $A_i b_{ij} \le B_j$, $1 \le j \le r$. Similarly, for every j, $1 \le j \le r$, we put

$$(0, 0, \ldots, 0, 1, 0, \ldots, 0)\theta^{-1} = (a_{j1}, a_{j2}, \ldots, a_{jk})$$
$$\Downarrow$$
$$j$$

where $a_{ji} \in A_i$, $1 \le i \le k$. Here we obtain the inclusion $B_j a_{ji} \le A_i$, $1 \le i \le k$.

Put $S = \| b_{ij} \|_{1 \le i \le k,\ 1 \le j \le r}$, $T = \| a_{ji} \|_{1 \le i \le k,\ 1 \le j \le r}$. For every i, $1 \le i \le k$, we have

$$(0, 0, \ldots, 0, 1, 0, \ldots, 0) = (0, 0, \ldots, 0, 1, 0, \ldots, 0)\theta\theta^{-1} =$$
$$(b_{i1}, b_{i2}, \ldots, b_{ir})\theta^{-1} = \sum_{1 \le j \le r}(0, 0, \ldots, 0, b_{ij}, 0, \ldots, 0)\theta^{-1} =$$
$$\sum_{1 \le j \le r}(b_{ij}a_{j1}, \ldots, b_{ij}a_{jk}) =$$
$$= (\sum_{1 \le j \le r} b_{ij}a_{j1}, \ldots, \sum_{1 \le j \le r} b_{ij}a_{jk}).$$

It follows that

$$\sum_{1 \le j \le r} b_{ij}a_{jt} = \begin{cases} 1, & \text{if } t = i; \\ 0, & \text{if } t \ne i. \end{cases}$$

This means that $ST = E_k$. Similarly, $TS = E_r$. Since coefficients of these matrixes belong to the field F, it is easy to show that $k = r$ and $T = S^{-1}$.

Put $A = A_1 \ldots A_k$ $B = B_1 \ldots B_k$. Then

$$A\,(\det S) = A(\sum_{\pi \in Sk} \operatorname{sign} \pi \cdot b_{1\pi(1)} \ldots b_{k\pi(k)}) =$$
$$\sum_{\pi \in Sk} \operatorname{sign} \pi \cdot A_1 b_{1\pi(1)} \ldots A_k b_{k\pi(k)} \le B_{\pi(1)} \ldots B_{\pi(k)} = B.$$

Similarly, $B \det T \le A$. Since $(\det S)(\det T) = 1$,

$$A = A\,(\det S)(\det T) \le B(\det T) \le A.$$

Hence $A = (\det T)B$; moreover, $\det T$ is a non – zero element of F.

4.25. Theorem. *Let D be a Dedekind domain, P be a projective D-module.*

(1) If P is finitely generated then $P \cong \underbrace{D \oplus \ldots \oplus D}_{k-1} \oplus L$ for some $k \ge 1$ and some non – zero ideal L of D. Furthermore, k is uniquely determined by P ($k = r_0(P)$) and L is determined up to multiplication by a non – zero constant in the field of fractions of D.

(2) If P is not finitely generated, then P is a free D – module.

Proof. **Corollary 4.21** implies that P is a direct sum of submodules, which are isomorphic with some ideals of D.

Suppose that P is finitely generated. By Lemma 4.24,

$$P \cong \underbrace{D \oplus \ldots \oplus D}_{k-1} \oplus L$$

for some non – zero ideal L. Corollary 4.6 shows that $k = r_0(P)$. Lemma 4.24 implies that L is determined up to multiplication by a non – zero element in the field of fractions of D.

Suppose that $P = \oplus_{\lambda \in \Lambda} R_\lambda$ where R_λ are isomorphic with some ideals of D, $\lambda \in \Lambda$, and the set Λ is infinite. Then we can suppose that

$$P \cong \oplus_{\lambda \in \Lambda} P_\lambda \text{ and } P_\lambda \cong \oplus_{n \in N} (A_{\lambda n} \oplus B_{\lambda n}),\ \lambda \in \Lambda,$$

where $A_{\lambda m}$ $B_{\lambda n}$ are non-zero ideals of D. Put $E_{\lambda 0} = D$, $E_{\lambda n} = A_{\lambda 1} B_{\lambda 1} \ldots A_{\lambda n} B_{\lambda n}$ $n \in \mathbf{N}$. Then $A_{\lambda n} B_{\lambda n} = E_{\lambda}^{-1}{}_{n-1} E_{\lambda n}$, $n \in \mathbf{N}$, $\lambda \in \Lambda$. So

$$A_{\lambda n} \oplus B_{\lambda n} \cong D \oplus A_{\lambda n} B_{\lambda n} = D \oplus E_{\lambda}^{-1}{}_{n-1} E_{\lambda n} \cong E_{\lambda}^{-1}{}_{n-1} \oplus E_{\lambda n}$$

by Lemma 4.24. We have

$$P_\lambda \cong E_{\lambda 0}^{-1} \oplus E_{\lambda 1} \oplus E_{\lambda 1}^{-1} \oplus E_{\lambda 2} \oplus \ldots$$

Since $E_{\lambda n} \oplus E_{\lambda n}^{-1} \cong D \oplus E_{\lambda n} E_{\lambda n}^{-1} = D \oplus D$, then P_λ is free for any $\lambda \in \Lambda$. Hence the module P is free.

Chapter 5

Injective Modules over Dedekind Domains

Let R be a ring, A and B be R– modules, $\phi: A \longrightarrow B$ be an R – homomorphism. If C is an R– module, then define the mapping $\Phi: \mathbf{Hom}_R(C, A) \longrightarrow \mathbf{Hom}_R(C, B)$ by the rule $(f)\Phi = f\phi$ for each $f \in \mathbf{Hom}_R(C, A)$. Note that $\mathbf{Hom}_R(C, B)$ and $\mathbf{Hom}_R(C, A)$ are abelian groups, i.e. they are $\mathbf{Z}$ – modules, and Φ is a $\mathbf{Z}$ – homomorphism. If the ring R is commutative, then Φ is an R – homomorphism.

5.1. Lemma. *Let R be a ring, A, B, C, E be R – modules. If*

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

is an exact sequence (ϕ and ψ are R – homomorphisms), then the sequence

$$0 \longrightarrow \mathbf{Hom}_R(E, A) \xrightarrow{\Phi} \mathbf{Hom}_R(E, B) \xrightarrow{\Psi} \mathbf{Hom}_R(E, C)$$

where $(f)\Phi = f\cdot\phi$, $(g)\Psi = g\cdot\psi$ for $f \in \mathbf{Hom}_R(E, A)$, $g \in \mathbf{Hom}_R(E, B)$, is likewise exact.

Proof. Since $Im\ \phi = Ker\ \psi$, we have $f(\Phi\Psi) = (f\phi)\psi = f(\phi\psi) = 0$ for each $f \in \mathbf{Hom}_R(E, A)$, that is, $Im\ \Phi \leq Ker\ \Psi$. Let $f \in Ker\ \Phi$, $e \in E$, then $0 = e(f\phi) = (ef)\phi$. Since ϕ is a monomorphism, $ef = 0$ for every $e \in E$, i.e. f is a zero homomorphism. So Φ is a monomorphism.

Let $g \in Ker\ \Psi$, that is, $g\psi = 0$. If $e \in E$, then $0 = e(g\psi) = (eg)\psi$, so $eg \in Ker\ \Psi = Im\ \phi$. Hence there exists an element $a \in A$ such that $eg = a\phi$. This element a is unique because ϕ is a monomorphism. We define now the mapping $g_1: E \longrightarrow A$ by the rule $eg_1 = a$. Then $e(g_1\phi) = (eg_1)\phi = a\phi = eg$ for every $e \in E$. This means that $g_1\phi = g$. Let $e_1, e_2 \in E$, $e_1g = a_1\phi$, $e_2g = a_2\phi$. Then

$$(e_1 + e_2)g = e_1g + e_2g = a_1\phi + a_2\phi = (a_1 + a_2)\phi.$$

This equation shows that

$$(e_1 + e_2)g = a_1 + a_2 = e_1g_1 + e_2g_1.$$

Similarly, we can prove that $(ex)g_1 = (eg_1)x$ for every $x \in R$. So $g = g_1\phi = g_1\Phi$ for some $g_1 \in \mathbf{Hom}_R(E, A)$, that is $g \in Im\ \Phi$.

Again let $\theta: A \longrightarrow B$ be an R – homomorphism. We define the mapping $\Theta: \mathbf{Hom}_R(B, C) \longrightarrow \mathbf{Hom}_R(A, C)$ by the rule $f\Theta = \theta f$ for every $f \in \mathbf{Hom}_R(B, C)$. Obviously, Θ is a $\mathbb{Z}$ – homomorphism. If the ring R is commutative, then Θ is an R– homomorphism.

5.2. Lemma. *Let R be a ring, A, B, C, E be R – modules. If*

$$A \xrightarrow{\theta} B \xrightarrow{\sigma} C \longrightarrow 0$$

is an exact sequence (here θ, σ are R – homomorphisms), then the sequence

$$0 \longrightarrow \mathbf{Hom}_R(C, E) \xrightarrow{\Sigma} \mathbf{Hom}_R(B, E) \xrightarrow{\Theta} \mathbf{Hom}_R(A, E)$$

where $f\Sigma = \sigma f$ and $g\Theta = \theta g$, $f \in \mathbf{Hom}_R(C, E)$, $g \in \mathbf{Hom}_R(B, E)$, is also an exact sequence.

Proof. Since $Im\ \theta = Ker\ \sigma$, we have

$$f(\Sigma\Theta) = (f\Sigma)\Theta = (\sigma f)\Theta = 0\cdot(\sigma f) = (\theta\sigma)f = 0$$

for each $f \in \mathbf{Hom}_R(C, E)$. Let $f \in Ker\ \Sigma$, $c \in C$. Then $0 = f\Sigma = \sigma f$. Since σ is an epimorphism, $c = b\sigma$ for suitable $b \in B$. Thus $cf = (b\ \sigma)f = b(\sigma f) = 0$ for any $c \in C$. This means that $f = 0$, so $\mathbf{Ker}\ \Sigma = < 0 >$ and Σ is a monomorphism. Let $g \in \mathbf{Ker}\ \Theta$, that is $\theta g = 0$, and $c \in C$. Since σ is an epimorphism, there exists an element $b \in B$ such that $c = b\sigma$. Furthermore, let b_1 be another element such that $c = b_1\sigma$. Then $b - b_1 \in \mathbf{Ker}\ \sigma = \mathbf{Im}\ \theta$, so there is an element $a \in A$ such that $b - b_1 = a\theta$. Then

$$0 = a(\theta g) = (a\theta)g = (b - b_1)y = bg - b_1g.$$

We define the mapping $g_1: C \longrightarrow E$ by the rule $cg_1 = bg$. From this definition it follows that $g = \sigma g_1 = g_1\Sigma$. As in the previous lemma, it is easy to show that g_1 is an R – homomorphism. So $\mathbf{Ker}\ \Theta = \mathbf{Im}\ \Sigma$.

A module E over a ring R is called *injective* if for every R– monomorphism $\beta: A \longrightarrow B$ and an R– homomorphism $\alpha: A \longrightarrow E$ there exists an R– homomorphism $\gamma: B \longrightarrow E$ such that $\beta\gamma = \alpha$.

5.3. Theorem (Baer's Criterion). *Let R be a ring, E be an R – module. Then E is an injective module if and only if for every right ideal J of R and for every R – homomorphism $\sigma: J \longrightarrow E$ there exists an R – homomorphism $\sigma^*: K \longrightarrow E$ such that $x\sigma^* = x\sigma$ for each $x \in J$.*

Proof. If E is an injective module, then we can consider J and R as R– modules. Let $\alpha: J \longrightarrow R$ be an identity embedding. Then there is an R– homomorphism $\sigma^*: R \longrightarrow E$ such that $\sigma = \alpha\sigma^*$. In particular, if $x \in J$, then

$$x\sigma = x(\alpha\sigma^*) = (x\alpha)\sigma^* = x\sigma^*.$$

Conversely, suppose that the conditions of this Theorem are valid. Let A, B be R – modules, $\alpha: A \longrightarrow E$ be a homomorphism. We can assume that A is an R – submodule of B.

Let

$$\mathcal{M} = \{ (C, \phi) \mid C \text{ be an } R\text{ – submodule satisfying } A \leq C \leq B \text{ and}$$
$$\phi: C \longrightarrow E \text{ be an } R\text{ – homomorphism such that } a\phi = a\alpha \text{ for each } a \in A \}.$$

Put $(C_1, \phi_1) \leq (C_2, \phi_2)$ if and only if $C_1 \leq C_2$ and the restriction of ϕ_2 on C_1 coincides with ϕ_1. So the set $\mathcal{M}$ is ordered.

Let $\mathcal{L}$ be a linearly ordered subset of $\mathcal{M}$, $C^{\wedge} = \cup_{(C,\ \phi) \in \mathcal{L}} C$. We define the mapping $\phi^{\#}: C^{\#} \longrightarrow E$ by the following rule. If $c \in C^{\#}$, then $c \in C$ where $(C, \phi) \in \mathcal{L}$. Put $c\phi^{\#} = c\phi$. If C_1 is another submodule with the properties $c \in C_1$ and $(C_1, \phi_1) \in \mathcal{L}$, then either $C \leq C_1$ (and in this case, $c\phi = c\phi_1$) or $C_1 \leq C$ (then $c\phi_1 = c\phi$). So the mapping $\phi^{\#}$ is defined correctly. It is easy to see that $\phi^{\#}$ is an R – homomorphism. So $(C, \phi) \leq (C^{\#}, \phi^{\#})$ for every $(C, \phi) \in \mathcal{L}$. Zorn's Lemma implies that the set $\mathcal{M}$ has a maximal element $(C, \bar{\phi})$.

Let $b \in B$, $J = \{ x \in K \mid bx \in \bar{C} \}$. Obviously, J is a right ideal of R. Define the mapping $\sigma: J \longrightarrow E$ by the rule $x\sigma = (bx)\ \bar{\phi}$. If $u \in R$, then

$$(xu)\ \sigma = (bxu)\ \bar{\phi} = (bx)\ \bar{\phi}\, u = (x\sigma)u.$$

Similarly, if $x_1, x_2 \in J$, then

$$(x_1 + x_2)\ \sigma = (b(x_1 + x_2))\ \bar{\phi} = (bx_1)\ \bar{\phi} + (bx_2)\ \bar{\phi} = x_1\sigma + x_2\sigma.$$

Thus σ is an R – homomorphism. Then there is an R – homomorphism $\sigma^*: R \longrightarrow E$, which extends σ. Define the mapping $\phi^*: \bar{C} + bK \longrightarrow E$ by the rule

$$(c + bx)\ \phi^* = c\ \bar{\phi} + x\ \sigma^*,\ c \in \bar{C},\ x \in K.$$

If $c + bx = c_1 + bx_1$ for some elements $c_1 \in \bar{C}$, $x_1 \in R$, then $b(x - x_1) = c_1 - c \in \bar{C}$, that is $x - x_1 \in J$. In this case we have

$$x\ \sigma^* - x_1\ \sigma^* = (x - x_1)\ \sigma^* = (x - x_1)\ \sigma = (b\ (x - x_1))\ \bar{\phi} = (c - c_1)\ \bar{\phi} = c_1\ \bar{\phi} - c\ \bar{\phi}.$$

It follows that $c\ \bar{\phi} + x\ \sigma^* = c_1\ \bar{\phi} + x_1\sigma^*$. So ϕ is a mapping. Furthermore

$$\begin{gathered}((c + bx) + (c_1 + bx_1))\ \bar{\phi} = ((c + c_1) + b(x + x_1))\ \bar{\phi}^* = \\ (c + c_1)\ \bar{\phi} + (x + x_1)\ \sigma^* = c\ \bar{\phi} + c_1\ \bar{\phi} + x\ \sigma^* + x_1\sigma^* = \\ c\ \bar{\phi} + x\ \sigma^* + c_1\ \bar{\phi} + x_1\sigma^* = (c + bx)\ \phi^* + (c_1 + bx_1)\ \bar{\phi}^*; \\ ((c + bx)y)\ \phi^* = (cy + b(xy))\ \phi^* = (c + bx)\ \bar{\phi}^* \cdot y\end{gathered}$$

for every $y \in R$. Hence $\bar{\phi}^*$ is an R – homomorphism. For every $c \in \bar{C}$ we have

$$c\ \bar{\phi} = (c + b \cdot 0)\ \bar{\phi}^* = c\ \bar{\phi} + 0\ \sigma^* = c\ \bar{\phi},$$

so $(\bar{C},\ \bar{\phi}) \leq (\bar{C} + bK,\ \bar{\phi}^*)$. This means that $\bar{C} + bK = \bar{C}$, because $(\bar{C},\ \bar{\phi})$ is a maximal element of $\mathcal{M}$. Hence $b \in \bar{C}$ and $\bar{C} = B$.

The next theorem provides us with another characterization of injective modules which is connected to exact sequences.

5.4. Theorem. *A R – module E is an R – injective if and only if the sequence*

$$0 \longrightarrow \mathbf{Hom}_R(C, E) \xrightarrow{\Sigma} \mathbf{Hom}_R(B, E) \xrightarrow{\Theta} \mathbf{Hom}_R(A, E) \longrightarrow 0$$

is exact for every exact sequence

$$0 \longrightarrow A \xrightarrow{\theta} B \xrightarrow{\sigma} C \longrightarrow 0$$

of R – modules and R – homomorphisms.

Proof. Let E be an injective module. Lemma 5.2 implies that the sequence

$$0 \longrightarrow \boldsymbol{Hom}_R(C, E) \xrightarrow{\Sigma} \boldsymbol{Hom}_R(B, E) \xrightarrow{\Theta} \boldsymbol{Hom}_R(A, E)$$

is exact. It is sufficient to show that Θ is an epimorphism. Let $f \in \boldsymbol{Hom}_R(A, E)$. Since Θ is a monomorphism, $f = \theta g$. In other words, $f = g\,\Theta$, so that Θ is an epimorphism.

Conversely, let $\theta: A \longrightarrow B$ be an R – monomorphism, $f: A \longrightarrow E$ be an R – homomorphism. Since the sequence

$$\boldsymbol{Hom}_R(B, E) \xrightarrow{\Theta} \boldsymbol{Hom}_R(A, E) \longrightarrow 0$$

is exact, Θ is an epimorphism. Thus $f = g\Theta$ for some $g \in \boldsymbol{Hom}_R(B, E)$. But $g\Theta = \theta g$, so $f = \theta g$. This means that E is an injective module.

Let R be an integral domain, A be an R – module, $0 \neq x \in R$. We say that A is x – *divisible* if for every element $a \in A$ there is an element $c \in A$ such that $cx = a$.

In other words, $A = Ax$.

The module A is called R – *divisible*, if A is x – divisible for every $0 \neq x \in R$.

Observe some properties of divisible modules.

5.5. Lemma. *Let R be an integral domain.*

(1) If E is an R – divisible module, C is an R – submodule of E, then E/C is an R – divisible module;

(2) If E_λ is an R – divisible module for every $\lambda \in \Lambda$, then the R – modules $\oplus_{\lambda \in \Lambda} E_\lambda$ and $\Pi_{\lambda \in \Lambda} E_\lambda$ are also R – divisible.

Proof. (1) Let $e \in E$, $x \in K$. There is an element $a \in E$ such that $ax = e$. Then $e + C = ax + C = (a + C)x$, that is $(E/C)x = E/C$.

(2) Let $(e_\lambda)_{\lambda \in \Lambda} \in \Pi_{\lambda \in \Lambda} E_\lambda$, $0 \neq x \in K$. There is an element $a_\lambda \in E_\lambda$ such that $a_\lambda x = e_\lambda$ for every $\lambda \in \Lambda$. Therefore $((a_\lambda)_{\lambda \in \Lambda})x = (e_\lambda)_{\lambda \in \Lambda}$, that is

$$(\Pi_{\lambda \in \Lambda} E_\lambda)\, x = \Pi_{\lambda \in \Lambda} E_\lambda$$

Suppose now that $\boldsymbol{Supp}((e_\lambda)_{\lambda \in \Lambda})$ is finite. Put

$$b_\lambda = \begin{cases} a_\lambda, \text{ if } \lambda \in \boldsymbol{Supp}((e_\lambda)_{\lambda \in \Lambda}); \\ 0, \text{ if } \lambda \notin \boldsymbol{Supp}((e_\lambda)_{\lambda \in \Lambda}). \end{cases}$$

Then $\boldsymbol{Supp}((b_\lambda)_{\lambda \in \Lambda}) = \boldsymbol{Supp}((e_\lambda)_{\lambda \in \Lambda})$, in particular, $(b_\lambda)_{\lambda \in \Lambda} \in \oplus_{\lambda \in \Lambda} E_\lambda$. Obviously, $((b_\lambda)_{\lambda \in \Lambda})\, x = (e_\lambda)_{\lambda \in \Lambda}$, hence $(\oplus_{\lambda \in \Lambda} E_\lambda)\, x = \oplus_{\lambda \in \Lambda} E_\lambda$

5.6. Proposition. *Let R be an integral domain, E be an injective R – module. Then E is an R – divisible module.*

Proof. Let $0 \neq x \in K$, $e \in E$. Define the mapping $f: Kx \longrightarrow E$ by the following rule $(ux)f = eu$, $u \in R$. Clearly, f is an R – homomorphism. Since E is an R – injective module, Theorem 5.3 implies that there is an R – homomorphism $f^*: R \longrightarrow E$ extending f. We have

$$e = xf = xf^* = (1\,x)f^* = (1 \cdot f^*)x. \text{ So } Ex = E,$$

i.e. E is an R – divisible module.

We can consider every abelian group as a $\mathbb{Z}$ – module.

An abelian group A is said to be *divisible* if A is a $\mathbb{Z}$ – divisible module (in other words, $A = kA$ for each $0 \neq k \in \mathbb{Z}$).

Proposition 5.6 has the following converse.

5.7. Proposition. *Let R be an integral domain, E be an R – module. If E is R – divisible and R – torsion – free, then E is an injective module.*

Proof. Let L be an ideal of R, $f: L \longrightarrow E$ be an R – homomorphism. We can assume that $L \neq < 0 >$ and choose an element $0 \neq x \in L$. Since E is R – divisible, there is an element $e \in E$ such that $ex = xf$. If $y \in L$, then

$$(yf)x = (yx)f = (xy)f = (xf)y = (ex)y = e(xy) = e(yx) = (ey)x.$$

It follows that $(yf - ey)x = 0$. This means that $yf = ey$, because E is R – torsion – free. Define the mapping $g: R \longrightarrow E$ by the following rule $ug = eu$ for every $u \in R$. Then $yg = ey = yf$ for each $y \in L$, so g is an extension of f. Theorem 5.3 implies that E is an injective module.

5.8. Proposition. *Let D be a Dedekind domain, E be a D – module. If E is a divisible module, then E is an injective module.*

Proof. Let L be an ideal of D. We can assume that $L \neq < 0 >$. Let $f: L \longrightarrow E$ be a D – homomorphism. We have $D = LL^{-1}$. Hence there are elements $x_1, \ldots, x_k \in L$, $y_1, \ldots, y_k \in L^{-1}$ such that $1 = x_1 y_1 + \ldots + x_k y_k$. We may assume that $x_j \neq 0$ for all j, $1 \leq j \leq k$. Since E is D – divisible, then there are elements $e_1, \ldots, e_k \in E$ such that $x_1 f = e_1 x_1, \ldots, x_k f = e_k x_k$. Let $x \in L$. Since $x y_j \in LL^{-1} = D$,

$$xf = (x\,1)f = (xx_1 y_1 + \ldots + xx_k y_k)f = (x_1 f)x y_1 + \ldots + (x_k f)x y_k = (e_1 x_1)xy_1 + \ldots + (e_k x_k)xy_k = (e_1 x_1 y_1 + \ldots + (e_k x_k y_k)x.$$

The inclusion $x_1 y_1 + \ldots + x_k y_k \in LL^{-1} = D$ implies that $e = e_1 x_1 y_1 + \ldots + e_k x_k y_k \in E$. Define the mapping $f^*: D \longrightarrow E$ by the rule $yf^* = ey$, $y \in D$. It is easy to see that f^* is a

D – homomorphism and $xf^* = e\,x = x\,f$, so f^* is an extension of f. Theorem 5.3 implies that E is an injective module.

5.9. Corollary. *Let R be a principal ideal domain, E be an R – module. Then E is injective if and only if E is R – divisible.*

5.10. Corollary (R. Baer). *An abelian group A is divisible if and only if A is an injective $\mathbb{Z}$ – module.*

5.11. Lemma. *Let D be a Dedekind domain, A be a D – module, $\{ E_\lambda \mid \lambda \in \Lambda \}$ be a family of submodules of A. If E_λ is D – divisible for any $\lambda \in \Lambda$, then $E = \sum_{\lambda \in \Lambda} E_\lambda$ is also D – divisible.*

Indeed, a sum of submodules is a homomorphic image of its direct sum, thus we may apply Lemma 5.5.

5.12. Corollary. *Let D be a Dedekind domain, A be a D – module. Then A includes the largest D – divisible submodule div(A).*

In fact, ***div(A)*** is the sum of all divisible submodules of A.

The submodule ***div(A)*** is called the D – ***divisible*** part of A.

5.13. Theorem. *Let D be a Dedekind domain, A be a D – module, E and B be submodules of A satisfying the following conditions:*

(1) E is a D – divisible submodule;

(2) $B \cap E = < 0 >$.

Then there exists a D – submodule C such that $B \leq C$ and $A = E \oplus C$.

Proof. Let j_1 be a canonical embedding $E \longrightarrow E \oplus B$, j_2 be a canonical embedding of $E \oplus B$ in A. The mapping α: $E + B \longrightarrow E$, defined by the rule $(e + b)\alpha = e$, $e \in E$, $b \in B$, is a D – homomorphism. Proposition 5.8 implies that E is a D – injective module. Hence there is a D – homomorphism β: $A \longrightarrow E$ such that $j_2\beta = \alpha$. Then $j_1 j_2 \beta = j_1 \alpha = \varepsilon_E$. Lemma 4.8 implies that $A = \mathbf{Im}\ (j_1 j_2) \oplus \mathbf{Ker}\ \beta$. Obviously, $\mathbf{Im}\ (j_1 j_2) = E$. Put $C = \mathbf{Ker}\ \beta$; then $A = E \oplus C$. If $b \in B$, then

$$b\beta = (bj_2)\beta = b\alpha = 0,$$

so $B \leq \mathbf{Ker}\ \beta = C$.

5.14. Corollary. *Let D be a Dedekind domain, A be a D – module. Then $A = div(A) \oplus B$, where B is a D – submodule and B does not include D – divisible submodules.*

Let F be the field of fractions of the Dedekind domain D. For all elements $a \in F$ and $0 \neq x \in D$ we have $a = (ax^{-1})x$. This means that the D – module F is divisible.

5.15. Theorem. *Let D be a Dedekind domain. Each D – module A can be imbedded in a divisible D – module.*

Proof. Proposition 4.9 implies that $A \cong_D C/B$ where $C = \bigoplus_{\lambda \in \Lambda} C_\lambda$, $C_\lambda \cong_D D$, B is a submodule of C. Every C_λ is imbedded in a D – module $E_\lambda \cong_D F$ where F is the field of fractions of D. Lemma 5.5 yields that the D – module $E^* = \bigoplus_{\lambda \in \Lambda} E_\lambda$ is D – divisible, and $E = E^*/B$ is D – divisible. Now we have $A \cong C/B \leq E^*/B = E$.

Let D be a Dedekind domain, A be a D – module. We say that a D – divisible module E is a *minimal divisible submodule for* A if every proper D – submodule B of E, including A, is not D – divisible.

5.16. Lemma. *Let D be a Dedekind domain, A be a D – module, E be a divisible module such that $A \leq E$. Then E includes a minimal divisible submodule for A. In particular, for every D – module A there exists a minimal divisible submodule.*

Proof. If every proper submodule B of E with the property $B \geq A$ is not divisible, then E is a minimal divisible module for A. Suppose that E includes a divisible submodule $E_1 \geq A$. By Theorem 5.13, $E = E_1 \oplus C$ for some submodule C. Since $C \cong E/E_1$, C is a divisible submodule by Lemma 5.5. Put

$$\mathfrak{M} = \{ C \mid C \text{ is a divisible submodule of } E \text{ such that } A \cap C = < 0 > \}.$$

Then $\mathfrak{M} \neq \varnothing$. Let $\mathfrak{L}$ be a linearly ordered (by inclusion) subset of $\mathfrak{M}$. $E_2 = \cup \mathfrak{L}$. Clearly, E_2 is a divisible submodule and $E_2 \cap A = < 0 >$. In other words, $E_2 \in \mathfrak{M}$. Zorn's Lemma implies that $\mathfrak{M}$ has a maximal element M. By Theorem 5.13, we obtain again that $E = M \oplus E_3$ where E_3 is D – submodule and $A \leq E_3$. Since $E_3 \cong E/M$, E_3 is divisible by Lemma 5.5. Theorem 5.13 implies the decomposition $E_3 = E_4 \oplus E_5$ where E_5 is a divisible submodule. Lemma 5.5 implies that $M \oplus E_5$ is divisible and $(M \oplus E_5) \cap A = < 0 >$. By the selection of M we obtain that $M = M \oplus E_5$, that is $E_5 = < 0 >$ and $E_3 = E_4$. This means that E_3 is a minimal divisible submodule for A.

If A is a D – module, then Theorem 5.15 implies that there exists a divisible module E with the property $E \geq A$.

5.17. Lemma. *Let D be a Dedekind domain, A be a D – module, $T = t_D(A)$. If A is a divisible module, then T is also divisible.*

Proof. Let $a \in T$, $0 \neq x \in D$, then $Ann_D(a) = J \neq < 0 >$. Since A is divisible, there is an element $b \in A$ such that $bx = a$. Let $0 \neq y \in J$. Since $xy \neq 0$ and

$$0 = ay = (bx) = b(xy),\ 0 \neq xy \in Ann_D(b),$$

that is, $b \in T$. Thus T is a divisible submodule.

5.18. Corollary. *Let D be a Dedekind domain, A be a D – module, E be a minimal divisible module for A.*

(1) If A is a D – periodic module, then E is also D – periodic.
(2) If A is D – torsion – free, then E is also D – torsion – free.

Proof. Let $T = t_D(E)$. By Lemma 5.17 T is divisible. If A is D – periodic, then $A \leq T$. The minimality of E guarantees that $E = T$. If A is D – torsion – free, then $A \cap T = < 0 >$. By Theorem 5.13, there exists a D – submodule $E_1 \geq A$ such that $E = E_1 \oplus T$. Since $E_1 \cong E/T$, E_1 is divisible by Lemma 5.5. Again the selection of E insure that $E = E_1$.

5.19. Lemma. *Let D be a Dedekind domain, $C = cD$ be a cyclic D – module, E be a minimal divisible submodule for C. If C is D – torsion – free, then $E \cong_D F$ where F is the field of fractions of ring D.*

Proof. Since C is D – torsion – free, the mapping $\phi: D \longrightarrow C$, defined by the rule $x\phi = cx$, $x \in D$, is a D – homomorphism. Corollary 5.18 implies that E is D – torsion – free. Let $c_1 \in C$, $0 \neq y \in D$, e_1, e_2 be elements of E such that $e_1y = c_1 = e_2y$. Then $0 = e_1y - e_2y = (e_1 - e_2)y$. Since E is D – torsion – free, $e_1 - e_2 = 0$ and $e_1 = e_2$. So for each $c_1 \in C$, $0 \neq y \in D$, the element e with the property $ey = c_1$ is unique. If $c_1 = cx$ for some $x \in D$, then put $e = c(x/y)$.

Extend the mapping ϕ to the mapping $\phi_1: F \longrightarrow E$ by the rule $(x/y)\phi_1 = c(x/y)$. Let $x_1 \in C$, $0 \neq y_1 \in D$, $e_1 = c(x_1/y_1)$. Then $ey = cx$, $e_1y_1 = cx_1$. We have

$$eyy_1 = cxy_1,\ e_1y_1y = cx_1y$$

and

$$(e + e_1)yy_1 = cx_1y + cxy_1 = c(x_1y + xy_1).$$

It follows that $e + e_1 = c\,((x_1y + xy_1)/yy_1)$. Hence

$$(x/y + x_1/y_1)\phi_1 = ((x_1y + xy_1)/yy_1)\phi_1 = c((x_1y + xy_1)/yy_1) = e + e_1 = c(x/y) + c(x_1/y_1) = (x/y)\phi_1 + (x_1/y_1)\phi_1.$$

Let $u \in D$. Put $e_2 = c(ux/y)$. Then $e_2y = cux - (cx)u = (ey)u = (eu)y$. It follows that $(e_2 - eu)y = 0$, so $e_2 = eu$. Thus

$$e_2 = (u(x/y))\phi_1 = eu = c(x/y)u = (x/y\phi_1 \cdot u.$$

This shows that ϕ_1 is a D – homomorphism. Let $x/y \in Ker\ \phi_1$, that is $c(x/y) = 0$. Then $cx = c(x/y)\,y = 0$, i.e. $x \in Ann_D(C) = < 0 >$. This means that $x/y = 0$ and $Ker\ \phi_1 = < 0 >$. Finally, $Im\ \phi_1$ is divisible submodule of E and $C \leq Im\ \phi_1$. By the choice of E, we obtain the equation $E = Im\ \phi_1$. Hence $E \cong F$.

Let D be a Dedekind domain, A be a simple D – module. Then $A \cong D/P$ for some maximal ideal P. Corollary 1.28 implies that D/P^k and P/P^{k+1} are isomorphic as D – modules for any $k \in \mathbb{N}$. In particular, the D – module D/P^k is embedded in the

D – module D/P^{k+1}, $k \in \mathbf{N}$. Therefore we can consider the injective limit of the family of D – modules $\{D/P^k \mid k \in \mathbf{N}\}$.

Put

$$\mathbf{C}_{P\infty} = \mathit{liminj}\{D/P^k \mid k \in \mathbf{N}\}.$$

The D – module $\mathbf{C}_{P\infty}$ is called the *Prüfer P – module.*

It follows from the construction, that $\mathbf{C}_{P\infty}$ is a P – module, moreover $\Omega_{P,k}(\mathbf{C}_{P\infty}) \cong D/P^k$, $k \in \mathbf{N}$. Furthermore,

$$\Omega_{P,k+1}(\mathbf{C}_{P\infty})/\Omega_{P,k}(\mathbf{C}_{P\infty}) \cong (D/P^{k+1})/(P/P^{k+1}) \cong D/P.$$

Hence, if C is a D – submodule of $\mathbf{C}_{P\infty}$ and $C \neq \mathbf{C}_{P\infty}$, then $C = \Omega_{P,k}(\mathbf{C}_{P\infty})$ for some $k \in \mathbf{N}$. Similarly, if $b \notin \Omega_{P,k-1}(\mathbf{C}_{P\infty})$, then $C = bD$.

Observe also that a Prüfer P – module is monolithic and its monolith coincides with $\Omega_{P,1}(\mathbf{C}_{P\infty})$.

5.20. Lemma. *A Prüfer P – module is divisible.*

Proof. Let $a \in \mathbf{C}_{P\infty}$, $0 \neq x \in D$. Then there is a number n such that $a \in \Omega_{P,n}(\mathbf{C}_{P\infty}) \setminus \Omega_{P,n-1}(\mathbf{C}_{P\infty})$. This means that $aD = \Omega_{P,n}(\mathbf{C}_{P\infty})$. Let $L = xD$. If P does not include L, then $L + P = D$. Lemma 1.18 implies that $L + P^n = D$, so that $1 = xu + y$, $u \in D$, $y \in P^n$. Then

$$a = a \cdot 1 = axu + ay = (au)x$$

because $\mathbf{Ann}_D(a) = P^n$.

Consider the case when $x \in P$. In Theorem 1.29 we have already proved that $\bigcap_{k \in N} P^k = <0>$. Therefore there exists a number t such that $x \in P^t/P^{t+1}$.

Consider the submodule $B = \Omega_{P,n+t}(\mathbf{C}_{P\infty})$. The mapping $\phi: b \longrightarrow bx$, $b \in B$, is a D – homomorphism; moreover, $Im\ \phi \leq \Omega_{P,n}(\mathbf{C}_{P\infty})$ because $x \in P^t$. If we assume that $Im\ \phi \leq \Omega_{P,n-1}(\mathbf{C}_{P\infty})$, then for any element $x_t \in P^{n-1}$, $b \in B$, we have $bxx_t = 0$. This means that $\mathbf{Ann}_D(B) \leq LP^{n-1} \leq P^t P^{n-1} = P^{t+n-1}$. However, $\mathbf{Ann}_D(B) = P^{n+t}$. This contradiction implies that $Im\ \phi \leq \Omega_{P,n-1}(\mathbf{C}_{P\infty})$, that is, $Im\ \phi = \Omega_{P,n}(\mathbf{C}_{P\infty}) = aD$. In particular, $a \in Im\ \phi$ and $a = bx$ for some element $b \in B$. So $\mathbf{C}_{P\infty}$ is a divisible module.

5.21. Corollary. *Let D be a Dedekind domain, A be a simple D – module, E be a minimal divisible module for A. Then $E \cong_D \mathbf{C}_{P\infty}$ where $P = \mathbf{Ann}_D(A)$.*

Proof. Corollary 5.18 implies that E is a D – periodic module. Then Corollary 3.8 yields that $E = \bigoplus_{Q \in \pi} E_Q$ where E_Q is the Q – component of E, $\pi = \mathbf{Ass}_D(A)$. In particular, $E = E_P \oplus C$ where C is the Q' – component, that is, $C = \bigoplus_{Q \neq P} E_Q$ The isomorphism $E_P \cong E/C$ together with Lemma 5.5 shows that E_P is a divisible module. From the choice of E we obtain that $E = E_P$, because $A \leq E_P$. So E is a P – module. Proposition 1.24 implies that $P = yD + P^2$ where $y \in P \setminus P^2$. Let $0 \neq a_1 \in A$. Then $A =$

a_1D because A is a simple module. Since E is a divisible module, there are elements a_n, $n \geq 2$ with the properties: $a_{n+1}x = a_n$, $n \in \mathbb{N}$. Obviously, $Ann_D(a_2) = Ann_D(a_2D) \neq P$, so that $Ann_D(a_2D) = P^{k_1}$ where $k_1 > 1$. Hence $a_2D \cong D/P^{k_1}$, in particular, $A_2 = a_2D$ includes submodules that are isomorphic to $D/P \cong P^{k_1-1}/P^{k_1} \cong A, \ldots, D/P^{k_1-1} \cong P/P^{k_1}$. Put $A_3 = a_3D$. Therefore $Ann_D(a_3) = Ann_D(A_3) = P^{k_2}$ and $k_2 > k_1$. Similarly, let $Ann(a_n) = P^{k_n-1}$, then $k_1 < k_2 < \ldots < k_n < \ldots$. Thus $\cup_{n \in \mathbb{N}} A_n$ includes a submodule B which is isomorphic to $\oplus_{i \in I}\{D/P^k \mid k \in \mathbb{N}\}$. Therefore $B = E$.

Let A and E be D – modules and $A \leq E$. We say that E is an *essential extension* of A (or A is an *essential submodule* of E) if for every non – zero submodule C of B the intersection $A \cap C$ is non – zero.

5.22. Lemma. *Let D be a Dedekind domain, A be a D - module, E be a divisible D-module and $A \leq E$. Then E is a minimal divisible module for A if and only if E is an essential extension of A.*

Proof. Let E be a minimal divisible module for A. Suppose that E includes a non – zero submodule B such that $B \cap A = <0>$.

Suppose first that B is D – torsion – free, and let $0 \neq b \in B$. Since E is a divisible module, E includes a minimal divisible submodule B_1 for bD by Lemma 5.16. Lemma 5.19 implies that $B_1 \cong_D F$ where F is the field of fractions of D. Since B_1/bD is D – periodic, $B_1 \cap A = <0>$. Using Theorem 5.13, we obtain that $E = E_1 \oplus B_1$ where E_1 is a D – submodule and $A \leq E_1$. By Lemma 5.5, $E/B_1 \cong E_1$ is a divisible submodule. However this contradicts to the choice of E.

Suppose that $t_D(B) \neq <0>$. Let $P \in Ass_D(B)$, B_P be the P – component of B, $0 \neq b \in B_P$. Then $bD \cong D/P^k$ for some $k \geq 1$. Therefore bD includes a D – submodule $B_2 \cong P^{k-1}/P^k \cong D/P$; thus it is a simple D – submodule. Lemma 5.16 implies that E includes the minimal divisible submodule B_2 for B_2, and $B_3 \cong \mathbb{C}_{p^\infty}$ by Corollary 5.21. In particular, B_3 is a monolithic D – module and $\mu_D(B_3) = B_2$. It follows that $B_3 \cap A = <0>$. Theorem 5.13 implies the decomposition $E = E_2 \oplus B_3$, where E_2 is a D – submodule such that $A \leq E_2$. Since E_2 is divisible by Lemma 5.5, we obtained a contradiction. It shows that E is an essential extension of A.

Conversely, let E be an essential extension of A and E be a divisible module. If E includes a divisible submodule $E_3 \geq A$, then Theorem 5.13 implies that $E = E_3 \oplus C$ for some D – submodule C. Since $C \cap A \leq C \cap E_3 = <0>$, $C = <0>$, that is, $E = E_3$.

5.23. Theorem. *Let D be a Dedekind domain, A be a D – module. Then there exists a minimal divisible module for A, and every two minimal divisible modules for A are isomorphic.*

Proof. Lemma 5.16 implies that there exists a minimal divisible module for A.

Let E_1 and E_2 be minimal divisible modules for A. Proposition 5.8 yields that E_2 is an injective D – module. Therefore for the identity automorphism $\varepsilon_A: A \longrightarrow A$ there is an extension $\theta: E_1 \longrightarrow E_2$, and $E_1\theta = E_2$, i.e. θ is an epimorphism. Assume that $Ker\ \theta \neq <0>$. Let $0 \neq a \in A \cap Ker\ \theta$. Then $0 = a\theta = a\varepsilon_A = a$. This contradiction shows that $Ker\ \theta = <0>$. So θ is an isomorphism.

A minimal divisible module for A we will call *a divisible (or injective) envelope* of module A.

5.24. Lemma. *Let R be a ring, $\{A_\lambda \mid \lambda \in \Lambda\}$ be a family of R-modules. If E_λ is an essential extension of A_λ for any $\lambda \in \Lambda$, then $\oplus_{\lambda \in \Lambda} E_\lambda$ is an essential extension of $\oplus_{\lambda \in \Lambda} A_\lambda$.*

Proof. Put $A = \oplus_{\lambda \in \Lambda} A_\lambda$, $E = \oplus_{\lambda \in \Lambda} E_\lambda$, $0 \neq e \in E$. Then $e = \sum_{\lambda \in \Lambda} e_\lambda$ where $\varsigma = Supp((e_\lambda)_{\lambda \in \Lambda})$ is finite. Let $\lambda \in \varsigma$, $0 \neq e_\lambda \in E_\lambda$. Since E_λ is an essential extension of A_λ, there is the element $x_1 \in R$ such that $0 \neq e_\lambda x_1 \in A_\lambda$. If $e_\mu x_1 = 0$ for $\mu \in \varsigma \setminus \{\lambda\}$, then $0 \neq ex_1 = e_\lambda x_1 \in A$, that is $eR \cap A \neq < 0 >$. Therefore we can assume that $e_\mu x_1 \neq 0$ for some $\mu \in \varsigma \setminus \{\lambda\}$. Since E_μ is an essential extension of A_μ, there is an element $x_2 \in R$ such that $0 \neq e_\mu x_1 x_2 \in A_\mu$. Note that $e_\lambda x_1 x_2 \in A_\lambda$. If $e_\nu x_1 x_2 = 0$ for $\nu \in \varsigma \setminus \{\lambda, \mu\}$, then

$$0 \neq ex_1x_n = e_\lambda x_1x_2 + e_\mu x_1x_2 \in A,$$

that is $eR \cap A \neq < 0 >$. Using similar arguments, after finitely many steps we obtain that $eR \cap A \neq < 0 >$. This means that E is an essential extension of A.

5.25. Corollary. *Let D be a Dedekind domain, $\{A_\lambda \mid \lambda \in \Lambda\}$ be a family of D-modules. If E_λ is a D-divisible envelope of A_λ, $\lambda \in \Lambda$, then $\oplus_{\lambda \in \Lambda} E_\lambda$ is a divisible envelope of $\oplus_{\lambda \in \Lambda} A_\lambda$.*

Proof. Lemma 5.5 implies that $U = \oplus_{\lambda \in \Lambda} E_\lambda$ is a divisible module. Lemma 5.16 proves that U includes a divisible envelope E of the module $A = \oplus_{\lambda \in \Lambda} A_\lambda$. By Theorem 5.13 $U = E \oplus B$ for some D-submodule B. In particular, $B \cap A = < 0 >$. Lemma 5.24 implies that U is an essential extension of A. This means that $B = < 0 >$, that is, $E = U$.

Let D be a Dedekind domain, A be a D-module. If P is a maximal ideal of D, then let A_P be a P-component of A. Since $Ann_D(\Omega_{P,1}(A_P)) = P$, we can consider $\Omega_{P,1}(A_P)$ as a vector space over the field D/P.

Put $r_P(A) = dim D/P(\Omega_{P,1}(A_P))$. $r_P(A)$ is called the *P-rank* of a module A.

Note that $r_P(A)$, as and $r_0(A)$, is an invariant of the module A.

5.26. Theorem *Let D be a Dedekind domain, E be a divisible D-module. Then $E = (\oplus_{P \in \pi} E_P) \oplus U$ where $\pi = Ass_D(E)$, $E_P = \oplus_{\lambda \in \Lambda(P)} C_\lambda$, C_λ is a Prüfer P-module for any $\lambda \in \Lambda(P)$; $U = \oplus_{\mu \in M} U_\mu$, $U_\mu \cong_D F$ where F is the field of fractions of the ring D. Moreover, $|\Lambda(P)| = r_P(E)$, $|M| = r_0(E)$.*

Proof. Let $T = t_D(E)$. Lemma 5.17 implies that T is divisible. By Theorem 5.13 $E = T \oplus U$ for some D-submodule U. Corollary 3.8 gives a decomposition $T = \oplus_{P \in \pi} E_P$ where E_P is a P – component of E, $\pi = Ass_D(E)$. By Lemma 5.5, E_P is divisible. We have $\Omega_{P,1}(E_P) = \oplus_{\lambda \in \Lambda(P)} M_\lambda$ where M_λ is a simple D-module, $M_\lambda \cong D/P$, $\lambda \in \Lambda(P)$. Using Lemma 5.16 we obtain that E_P includes a divisible envelope C of $\Omega_{P,1}(E_P)$. Corollary 5.25 implies that $C = \oplus_{\lambda \in \Lambda(P)} C_\lambda$ where C_λ is a divisible envelope of M_λ, $\lambda \in \Lambda(P)$. By Corollary 5.21, C_λ is a Prüfer P-module, $\lambda \in \Lambda(P)$. By Theorem 5.13, there exists a submodule A such that $E_P = C \oplus A$. Since $\Omega_{P,1}(E_P) \leq C$ and E_P is an essential extension of $\Omega_{P,1}(E_P)$, $A = < 0 >$, that is, $E_P = C$. By the definition of $\Lambda(P)$, we obtain that $|\Lambda(P)| = r_P(A)$.

The isomorphism $U \cong E/T$ implies that $r_0(E) = r_0(U)$. Let S be a maximal D – free subset of U, $V = \oplus_{a \in S} aD$. If $u \in U$, then $uD \cap V \neq < 0 >$, so U is an essential extension of V. Since $r_0(U) = |S|$, $r_0(E) = |S|$.

By $C \cong E/T$, from Lemma 5.5 it follows that U is divisible. Lemma 5.16 implies that U includes a D – divisible envelope W of V. Corollary 5.25 shows that $W = \oplus_{a \in S} V_a$ where V_a is a D – divisible envelope of aD. By Lemma 5.19, $V_a \cong_D F$ where F is the field of fractions of D. By Theorem 5.13, there exists a submodule M such that $U = W \oplus M$. Since $V \leq W$ and U is an essential extension of V, $M = < 0 >$, and so $U = W$.

5.27. Corollary. *Let D be a Dedekind domain, A be a D – module, E be a divisible envelope of A.*

(1) $E = (\oplus_{P \in \pi} E_P) \oplus U$ where $\pi = Ass_D(E)$, $E_P = \oplus_{\lambda \in \Lambda(P)} C_\lambda$, C_λ is a Prüfer P – module for any $\lambda \in \Lambda(P)$;

(2) $U = \oplus_{\mu \in M} U_\mu$, $U_\mu \cong_D F$ where F is the field of fractions of the ring D;

(3) $|\Lambda(P)| = r_P(A)$, $|M| = r_0(A)$.

Finally, we can formulate another characterization of Dedekind domains, which is connected to the concepts of divisibility and injectivity.

5.28. Theorem *Let R be an integral domain. The following statements are equivalent:*

(1) R is a Dedekind domain;

(2) every divisible R – module is injective.

The proof requires some specific results from the theory of commutative rings and we omit it. One can find it, for example, in the book [SV1972, Theorem 4.25].

Chapter



Direct Sums of cyclic Modules

The Prüfer's and Kulikov's theorems play a very important role in the Abelian Group Theory. They provide us with the conditions for decomposability of abelian group into direct sums of cyclic subgroups. In this chapter we will consider their analogies for modules over Dedekind domains.

6.1. Proposition *Let D be a Dedekind domain, P a maximal ideal of D, $y \in P \setminus P^2$. A a P – module over the ring D. Then A is D – divisible if and only if $A = Ay$.*

Proof. If A is divisible, then $A = Ay$. Conversely, assume that $A = Ay$. Let $a \in A$, $0 \neq x \in D$. If $x \notin P$; then $xD + P = D$. Since A is a P – module, $Ann_D(a) = P^s$ for some $s \in \mathbb{N}$. By Lemma 1.18, $xD + P^s = D$, so that $1 = xu + y$ where $u \in D$, $y \in P^s$. We have now

$$a = a\,1 = a(xu + y) = axu + ay = aux = (au)x.$$

Suppose that $x \in P$. The equation $A = Ay$ implies that $A = Ay^n$ for each $n \in \mathbf{N}$. By Corollary 1.13 and Lemmas 1.18, 1.19, $xD = P^l Q$ where $P + Q = D$. It follows that $x \in P^l \setminus P^{l+1}$, so $x = x_1 z$ where $x_1 \in P^l$, $z \in Q$. Moreover, $z \notin P$. Using the equation $A = Ay^l$ we obtain that $a = a_1 y^l$ for some element $a_1 \in A$. Since $Ann_D(a) = P^s$, we have $P^{s+l} \leq Ann_D(a_1)$. Proposition 1.24 yields that $P^l = Dy^l + P^{l+s}$, so $x_1 = y^l u + v$, $u \in D$, $v \in P^{l-s}$. From $x \notin P^{l+1}$ it follows that $u \notin P$. Since P is a prime ideal, $u \notin P$ and $z \notin P$ we can conclude that $uz \notin P$. Hence $uzP + P = D$. This implies that $uzD + P^{l+s} = D$, so $1 = uzx_2 + x_3$ where $x_2 \in D$, $x_3 \in P^{l+s}$. Then

$$a_1 = a_1 1 = a_1(uzx_2 + x_3) = (a_1 x_2)uz + a_1 x_3 = a_2 uz,$$

where $a_2 = a_1 x_2$. In particular, $Ann_D(a_1) \leq Ann_D(a_2)$, that is, $P^{l+s} \leq Ann_D(a_2)$. It follows that

$$a = a_1 y^l = (a_2 uz)y^l = a_2 uz\, y^l + a_2 vz = a_2 (uz\, y^l + vz) = a_2 x, \text{ so } A = Ax.$$

6.2. Proposition. *Let D be a Dedekind domain, P a maximal ideal of D, $y \in P \setminus P^2$, A a P–module over a ring D. Then $AP^s = Ay^s$ for every $s \in \mathbf{N}$.*

Proof. Let $a \in AP$, then $a = a_1 x_1 + \ldots + a_n x_n$ where $a_i \in A$, $x_j \in P$, $1 \leq j \leq n$. Since A is a P–module, $Ann_D(x_j) = P^{k_j}$ for some $k_j \in \mathbf{N}$, $1 \leq j \leq n$. Let $k = max\{k_1, \ldots, k_n\}$; then $a_i P^k = < 0 >$ for each j, $1 \leq j \leq n$. Proposition 1.24 implies that $P = yD + P^k$, so that $x_j = y u_j + v_j$ where $u_j \in D$, $v_j \in P^k$, $1 \leq j \leq n$. We have

$$a = \Sigma_{1 \leq j \leq n} a_j x_j = \Sigma_{1 \leq j \leq n} a_j (yu_j + v_j) =$$
$$\Sigma_{1 \leq j \leq n} (a_j yu_j + a_j v_j) = \Sigma_{1 \leq j \leq n} (a_j u_j)y = cy,$$

where $c = \Sigma_{1 \leq j \leq n} a_j u_j$. This means that $a \in Ay$, that is $AP = Ay$.

Let $s > 1$. We have already proved that $AP^s = Ay^s$. Put $B = AP^s$. Then

$$AP^{s+1} = (AP^s)P = BP = By = (Ay^s)y = Ay^{s+1}.$$

6.3. Theorem. *Let D be a Dedekind domain, $P \in Spec(D)$, A a P–module over ring D. Then A is decomposable in a direct sum of cyclic submodules if and only if there are the family of submodules $\{ A_n \mid n \in \mathbf{N} \}$ and the family of numbers $\{ k_n \mid n \in \mathbf{N} \}$ with the following properties:*

(1) $A_n \leq A_{n+1}$ for each $n \in \mathbf{N}$;

(2) $A = \bigcup_{n \in N} A_n$;

(3) $A_n \cap A P^k{}_n = < 0 >$, $n \in \mathbf{N}$.

Proof. Without loss of generality we can assume that $A_n \cap AP^n = < 0 >$ for each

$n \in \mathbf{N}$. Let $\mathcal{G}$ be the set of chains of submodules $\{G_n \mid n \in \mathbf{N}\}$ satisfying the following conditions:

$$G_n \leq G_{n+1},\ A_n \leq G_n,\ G_n \cap A P^n = <0>,\ n \in \mathbf{N}.$$

If $\{G_n \mid n \in \mathbf{N}\}, \{H_n \mid n \in \mathbf{N}\} \in \mathcal{G}$, then we put

$$\{G_n \mid n \in \mathbf{N}\} \ll \{H_n \mid n \in \mathbf{N}\}$$ if and only if $G_n \leq H_n$ for every $n \in \mathbf{N}$.

It is easy to see, that this relation is orderliness. Furthermore, if $\mathcal{G}$ is the linearly ordered subset of $\mathcal{G}$, $\cup \mathcal{G} \in \mathcal{G}$. Zorn's Lemma implies that $\mathcal{G}$ has a maximal element $\{M_n \mid n \in \mathbf{N}\}$.

Let $y \in P \setminus P^2$; then $AP^n = Ay^n$ for any $n \in \mathbf{N}$ by Proposition 6.2. We consider the submodule $\Omega_{P,1}(M_n) \cap AP^{n-1}$ as a vector space over the field D/P. Let $\{c_\lambda \mid \lambda \in \Lambda_n\}$ be a basis of this vector space. Then $\Omega_{P,1}(M_n) \cap AP^{n-1} = \oplus_{\lambda \in \Lambda_n} c_\lambda D$.

Put $\Lambda = \cup_{n \in N} \Lambda_n$, $C = \sum_{\lambda \in \Lambda} c_\lambda D$. Since $\sum_{\lambda \in \Lambda_1 \cup \ldots \cup \Lambda_n} c_\lambda D \leq M_n$ and $\oplus_{\lambda \in \Lambda_{n+1}} c_\lambda D \leq \Omega_{P,1}(M_{n+1}) \cap AP^n$, we obtain that

$$\sum_{\lambda \in \Lambda_1 \cup \ldots \cup \Lambda_n} c_\lambda D \cap \oplus_{\lambda \in \Lambda_{n+1}} c_\lambda D = <0>.$$

It implies that $\sum_{\lambda \in \Lambda} c_\lambda D = \oplus_{\lambda \in \Lambda} c_\lambda D$.

We will show that $\Omega_{P,1}(A) \cap M_r \leq C$ for every $r \in \mathbf{N}$. For $r = 1$ it is obvious. Assume that we have already proved the inclusion $\Omega_{P,1}(A) \cap M_r \leq C$ and let $b \in \Omega_{P,1}(M_{r+1}) \setminus M_r$. In particular, $(M_r + bD) \cap Ay^r \neq <0>$. Let $0 \neq g + bx = c \in Ay^r$, $g \in M_r$. Since $Ann_D(b) = P$, $x \in P$, that is, $xD + P = D$ and $1 = x x_1 + x_2$ for some elements $x_1 \in D$, $x_2 \in P$. It follows that

$$b = b\,1 = b(x x_1 + x_2) = (bx) x_1 + b x_2 = bx x_1.$$

Moreover, $x_1 \notin P$. Now we have $0 \neq g x_1 + bx x_1 = c x_1 \in Ay^r$. Put $g_1 = g x_1$, $c_1 = c x_1$; then $0 \neq g_1 + b = c_1 \in Ay^r$ and $c_1 \in Ay^r \cap M_{r+1}$. Suppose that $c_1 y \neq 0$. It follows that $c_1 y = g_1 y \neq 0$. There exists a number t such that $0 \neq (g_1 y) y^{t-1} \in \Omega_{P,1}(M_r)$, so

$$0 \neq c_1 y^t \in Ay^r \cap \Omega_{P,1}(M_r) \leq M_r \cap Ay^r = <0>.$$

This contradiction shows that $c_1 y = 0$. So $c_1 \in A_r \cap \Omega_{P,1}(M_{r+1}) = \oplus_{\lambda \in \Lambda_{r+1}} c_\lambda D \leq C$. Furthermore, $g_1 = c_1 - b$ and $g_1 y = c_1 y - by = 0$, so that $g \in \Omega_{P,1}(A) \cap M_r \leq C$. The equation $b = c_1 - g_1$ implies that $b \in C$. Thus $\Omega_{P,1}(A) \cap M_{r+1} \leq C$. Since $A = \cup_{n \in N} A_n$, $\Omega_{P,1}(A) = C$.

If $\lambda \in \Lambda_1$, then put $a_\lambda = c_\lambda$. Let $\lambda \in \Lambda_2$; then $c_\lambda \in Ay$, so there is an element $a_\lambda \in A$ such that $c_\lambda = a_\lambda y$. Let we have already chosen the elements a_λ for

$\lambda \in \Lambda_1 \cup \ldots \cup \Lambda_n$ and $\lambda \in \Lambda_{n+1}$. Then $c_\lambda \in Ay^n$. Hence there is an element a_λ such that $a_\lambda y^n = c_\lambda$.

Assume that $\sum_{\lambda \in \Lambda} a_\lambda D$ is not direct. Then there are the indexes $\mu(1), \ldots, \mu(m) \in \Lambda$ such that $a_{\mu(1)} x_1 + \ldots + a_{\mu(m)} x_m = 0$ for some $x_1, \ldots, x_m \in D$ where $a_{\mu(j)} x_j \neq 0$ for every j, $1 \leq j \leq m$. Choose the numbers $k_1, \ldots, k_m \in \mathbb{N}$ such that $a_{\mu(j)} y^{k_j} = c_{\mu(j)}$, $1 \leq j \leq m$. By Corollary 6.13 and Lemmas 1.18, 1.19, $x_j D = P^{r_j} Q_j$ where $P + Q_j = D$, $1 \leq j \leq m$. By the selection of elements $a_{\mu(j)}$ we obtain that $P^{k_j+1} \leq Ann_D(a_{\mu(j)})$, in particular, $r_j < k_j + 1$, $1 \leq j \leq m$. Put $s = max\{ k_j \mid 1 \leq j \leq m \}$. Proposition 1.24 implies that $P = yD + P^{s+1}$ and $P^{r_j} = y^{r_j} D + P^{s+1}$, so

$$x_j = (u_j y^{r_j} + v_j)z_j,\ u_j \in D,\ v_j \in P^{s+1},\ z_j \in Q_j,\ 1 \leq j \leq m.$$

If we assume that $u_j z_j \in P$, then $x_j \in P^{r_j+1}$. It follows that $x_j D = P^{r_j+1} L$ for some ideal L. However, this contradicts Theorem 1.14. This contradiction shows that $u_j z_j \notin P$, $1 \leq j \leq m$.

Let d be a number such that $k_d - r_d \geq k_j - r_j$ for every j, $1 \leq j \leq m$. Without loss of generality we can assume that $d = 1$. Then

$$a_{\mu(1)} x_1 y^{k_1 - r_1} = a_{\mu(1)} u_1 z_1 y^{r_1} y^{k_1 - r_1} + a_{\mu(1)} v_1 z_1 y^{k_1 - r_1} = a_{\mu(1)} u_1 z_1 y^{k_1} = c_{\mu(1)} u_1 z_1.$$

Since $u_1 z_1 \notin P$, $c_{\mu(1)} u_1 z_1 \neq 0$. Furthermore,

$$a_{\mu(j)} x_j y^{k_1 - r_1} = a_{\mu(j)} u_j z_j y^{r_j} y^{k_1 - r_1} + a_{\mu(j)} v_j z_j y^{k_1 - r_1} = a_{\mu(j)} u_j z_j y^{r_j + k_1 - r_1}.$$

Since $k_1 - r_1 \geq k_j - r_j$, $k_1 + r_j - r_1 \geq k_j$, therefore $a_{\mu(j)} x_j y^{k_1 - r_1} = \in c_{\mu(j)} D$, $2 \leq j \leq m$. Hence we obtain the equation $c_{\mu(1)} u_1 z_1 + c_{\mu(2)} h_2 + \ldots + c_{\mu(m)} h_m = 0$ for some $h_2, \ldots, h_m \in D$ where $c_{\mu(1)} u_1 z_1 \neq 0$. However this contradics the equation $\sum_{\lambda \in \Lambda} c_\lambda D = \bigoplus_{\lambda \in \Lambda} c_\lambda D$. Consequently, we prove the equation $\sum_{\lambda \in \Lambda} a_\lambda D = \bigoplus_{\lambda \in \Lambda} a_\lambda D$.

We have already proved the inclusion $\Omega_{P,1}(A) \leq C \leq \bigoplus_{\lambda \in \Lambda} a_\lambda D$. Assume that we have already proved the inclusion $\Omega_{P,n}(A) \leq \bigoplus_{\lambda \in \Lambda} a_\lambda D$ and let $b \in \Omega_{P,n+1}(A)$. Then $b y^n \in \Omega_{P,1}(A)$, so $b y^n = c_{\nu(1)} w_1 + \ldots + c_{\nu(q)} w_q$ for some $w_1, \ldots, w_q \in D$. We can assume that $\nu(1), \ldots, \nu(t) \in \Lambda_1 \cup \ldots \cup \Lambda_{n-1}$, $\nu(t+1), \ldots, \nu(q) \in \bigcup_{j \geq n} \Lambda_j$. In particular, $c_{\nu(j)} w_j = a_{\nu(j)} y^n g_j$ for $t+1 \leq j \leq q$. Therefore,

$$(b - a_{\nu(t+1)} g_{t+1} - \ldots - a_{\nu(q)} g_q) y^n = c_{\nu(1)} w_1 + \ldots + c_{\nu(t)} w_t \in M_{n-1}.$$

This equation implies that $c_{\nu(1)} w_1 + \ldots + c_{\nu(t)} w_t \in M_{n-1} \cap Ay^n = < 0 >$. This means that

$$(b - a_{\nu(t+1)} g_{t+1} - \ldots - a_{\nu(q)} g_q) \in \Omega_{P,n}(A) \leq \bigoplus_{\lambda \in \Lambda} a_\lambda D.$$

Hence $b \in \oplus_{\lambda \in \Lambda} a_\lambda D$. In other words, $\Omega_{P,\, n+1}(A) \leq \oplus_{\lambda \in \Lambda} a_\lambda D$. Since $A = \cup_{n \in N} \Omega_{P,\, n}(A)$, we obtain finally

$$A = \oplus_{\lambda \in \Lambda} a_\lambda D.$$

Conversely, let $A = \oplus_{\lambda \in \Lambda} a_\lambda D$. Put

$$\Lambda_n = \{ \lambda \in \Lambda \mid P^n = Ann_D(a_\lambda) \},\ A_n = \oplus_{\lambda \in \Lambda n} a_\lambda D,\ n \in \mathbf{N}.$$

Obviously, $\Lambda_n \subseteq \Lambda_{n+1}$, so $A_n \leq A_{n+1}$ for all $n \in \mathbf{N}$. Since A is a P – module, for each $\lambda \in \Lambda$ there is a number n_λ such that $P^{n_\lambda} \leq Ann_D(a_\lambda)$. Thus, $A = \cup_{n \in N} A_n$. Furthermore, $A = A_n \oplus B_n$ where $B_n = \oplus_{\lambda \in \Lambda \setminus \Lambda n} a_\lambda D$. Since $P^n \leq Ann_D(A_n)$, $AP^n \leq B_n$. Consequently, $A_n \cap AP^n = < 0 >$ for every $n \in \mathbf{N}$.

Observe that Theorem 6.3 is an analog of the famous Kulikov's criterion for abelian groups (see, for example, [FL1970, Theorem 17,2]).

6.4. Corollary *Let D be a Dedekind domain, A a P-module over the ring D, P a maximal ideal of D, B a submodule of A. If A is decomposable into the direct sum of cyclic submodules, then B is decomposable into the direct sum of cyclic submodules.*

Proof. By Theorem 6.3 there are the family of submodules $\{ A_n \mid n \in \mathbf{N} \}$ and the family of numbers $\{ k_n \mid n \in \mathbf{N} \}$ satisfying the following properties:

$$A = \cup_{n \in N} A_n;\ A_n \cap A P^{k_n} = < 0 > \text{ and } A_n \leq A_{n+1} \text{ for each } n \in \mathbf{N}.$$

Put $B_n = A_n \cap B$, $n \in \mathbf{N}$. Then $B_n \leq B_{n+1}$ and $B = \cup_{n \in N} B_n$. Furthermore,

$$B_n \cap B P^{k_n} \leq A_n \cap A P^{k_n} = < 0 >.$$

So we can apply Theorem 6.3 to the submodule B.

6.5. Corollary. *Let D be a Dedekind domain, A a D – module. If $Ann_D(A) \neq <0>$, then A is a direct sum of cyclic submodules.*

Proof. By Corollary 3.8 $A = \oplus_{P \in \pi} A_P$ where A_P is the P – component of A, $\pi = Ass_D(A)$. Corollary 6.13 implies that $Ann_D(A) = P_1^{k_1} \ldots P_m^{k_m}$ for some $P_1, \ldots, P_m \in Spec(D)$. It follow that $Ass_D(A) = \{ P_1, \ldots, P_m \}$. Applying Theorem 6.3 to every P_j – component of A, we prove this result.

This statement is an analog of the first Prüfer's theorem (see, for example, [FL1970, Theorem 17.2]).

6.6. Corollary. *Let D be a Dedekind domain, A a countably generated P – module over D, $P \in Spec(D)$. Then the module A is decomposable into a direct sum of cyclic submodules if and only if $\bigcap_{n \in N} AP^n = < 0 >$.*

Proof. Let A be generated by the subset $\{ a_n \mid n \in \mathbf{N} \}$. Put $A_n = a_1 D + \ldots + a_n D$. Since A is a D – periodic module, $Ann_D(A_n) \neq < 0 >$. Then A_n has a finite composition series. Let $Soc_D(A_n) = \oplus_{\lambda \in \Phi_n} M_\lambda$ where M_λ is a simple submodule, $\lambda \in \Phi_n$. If $0 \neq a \in M_\lambda$, then there is a number s_λ such that $a \in AP^{s_\lambda}$. Since $aD = M_\lambda$, it follows that $M_\lambda \cap AP^{s_\lambda} = < 0 >$. Let $k_n = max\{s_\lambda \mid \lambda \in \Phi_n\}$. Then $Soc_D(A_n) \cap AP^{k_n} = < 0 >$. This means that $A_n \cap AP^{k_n} = < 0 >$. Since $A_n \leq A_{n+1}$ and $\bigcup_{n \in N} A_n = A$, all conditions of Theorem 6.3 are realized. Hence A is a direct sum of cyclic submodules.

6.7. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A a D – periodic module. If $P \notin Ass_D(A)$, then $A = AP$.*

Proof. Corollary 3.8 yields that $A = \oplus_{Q \in \pi} A_Q$ where A_Q is the Q – component of A, $\pi = Ass_D(A)$. Therefore it is sufficient to prove that $(A_Q)P = A_Q$. If $a \in A_Q$, then $Ann_D(a) = Q^k$ for some $k \in \mathbf{N}$. Let $x \in P \setminus Q$. Since Q is a maximal ideal, $xD + Q = D$. Lemma 6.18 implies that $xD + Q^k = D$, thus $1 = xu + v$ where $u \in D$, $v \in Q^k$. We have

$$a = a\,1 = aux + av = (au)x \in (A_Q)x \leq (A_Q)P.$$

Consequently, $(A_Q)P = A_Q$ for every $Q \in Ass_D(A)$. Hence $AP = A$.

6.8. Corollary. *Let D be a Dedekind domain, A a D – periodic module, L an ideal of D, $L = P_1^{k_1} \ldots P_s^{k_s} Q_1^{m_1} \ldots Q_r^{m_r}$, where $P_1, \ldots, P_s, Q_1, \ldots, Q_r \in Spec(D)$, $P_i \neq Q_j \neq Q_t$ for $i \neq j \neq t$, $P_1, \ldots, P_s \in Ass_D(A)$, $Q_1, \ldots, Q_r \notin Ass_D(A)$. Then $AL = A P_1^{k_1} \ldots P_s^{k_s}$.*

6.9. Corollary. *Let D be a Dedekind domain, A a periodic countably generated D-module, $\mathfrak{B}$ be a family of all non – zero ideal of D. Then the module A is decomposable into a direct sum of cyclic submodules if and only if $\bigcap \mathfrak{B} = < 0 >$.*

Proof. Corollary 3.8 yields that $A = \oplus_{Q \in \pi} A_Q$ where A_Q is the Q – component of A, $\pi = Ass_D(A)$. Choose an arbitrary $P \in Ass_D(A)$. Let L be a non – zero ideal of D. By Corollary 1.13, $L = P^k Q_1^{m_1} \ldots Q_r^{m_r}$, where $Q_1, \ldots, Q_r \in Spec(D)$, $P \neq Q_j \neq Q_i$ for $i \neq j \neq t$. Corollary 6.8 shows that $A_P L = A_P P^k$. It follows that $\bigcap_{k \in N} A_P P^k = < 0 >$. Now we can apply Corollary 6.6.

Corollary 6.9 is an analog of the second Prüfer's Theorem (see, for example, [FL1970, Theorem 17.3]).

Now we must consider the question on the uniqueness of this direct decomposition.

6.10. Theorem. *Let D be a Dedekind domain, $P \in Spec(D)$, A be a P – module over D, $A = \oplus_{\lambda \in \Lambda} a_\lambda D = \oplus_{\mu \in M} b_\mu D$. Then there are an automorphism ϕ of A and a bijection $\theta: M \longrightarrow \Lambda$ such that $b_\mu \phi = a_{\mu\theta}$, $\mu \in M$.*

Proof. Put $\Lambda_n = \{ \lambda \in \Lambda \mid P^n = Ann_D(a_\lambda) \}$, $M_n = \{ \mu \in M \mid P^n = Ann_D(b_\mu) \}$. It is sufficient to show that $|\Lambda_n| = |M_n|$ for all $n \in \mathbf{N}$. Put $C_n = \Omega_{P,1}(A) \cap AP^{n-1}$. We have

$$AP^{n-1} = (\oplus_{\lambda \in \Lambda_n} a_\lambda P^{n-1}) \oplus (\oplus_{\lambda \in \Lambda \setminus \Delta} a_\lambda P^{n-1}),$$
$$AP^{n-1} = (\oplus_{\mu \in M_n} b_\mu P^{n-1}) \oplus (\oplus_{\mu \in M \setminus \Gamma} b_\mu P^{n-1})$$

where $\Delta = \Lambda_1 \cup \ldots \cup \Lambda_n$, $\Gamma = M_1 \cup \ldots \cup M_n$.

Thus

$$C_n = (\oplus_{\lambda \in \Lambda_n} a_\lambda P^{n-1}) \oplus C_{n+1} = (\oplus_{\mu \in M_n} b_\mu P^{n-1}) \oplus C_{n+1}.$$

It follows that $|\Lambda_n| = |M_n| = dim_{D/P}(C_{n+1}/C_n)$.

The following theorem shows that Corollary 6.5 is a characteristics of Dedekind domains (see, for example, [SV1972, Theorem 6.14]).

6.11. Theorem. *Let R be an integral domain. The following assertions are equivalent:*

(1) R is a Dedekind domain;

(2) if A is an R – module with $Ann_D(A) \neq < 0 >$, then A is a direct sum of cyclic submodules.

Chapter



Minimax Modules over Dedekind Domains

Let $(\mathcal{P}, \leq)$ be an ordered set $\mathcal{P}$ We say that $\mathcal{P}$ satisfies *the minimal condition* if every non – empty subset of $\mathcal{P}$ has a minimal element. This is equivalent to $\mathcal{P}$ satisfies the *descending chain condition*, that is, given a descending chain

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots,$$

of elements of $\mathcal{P}$, there is some $k \in \mathbf{N}$ such that $a_k = a_{k+n}$ for any $n \in \mathbf{N}$.

We say that a module A over a ring R is *an artinian* R – module or A satisfies *the minimal condition on its submodules* if $\mathcal{P}_R(A)$ satisfies the minimal condition. As we mentioned above, this is equivalent to the members of $\mathcal{P}_R(A)$ satisfying the descending chain condition.

Observe some elementary properties of artinian modules.

7.1. Lemma. *Let R be a ring and A an R – module.*

(1) If A is artinian and B a submodule of A, then B and A/B are artinian.

(2) If B is a submodule of A such that B and A/B are artinian, then A is artinian.

(3) If A has a finite series of submodules

$$<0> = A_0 \leq A_1 \leq \ldots \leq A_n = A,$$

every factor of which is artinian then A is likewise artinian.

(4) If $A = A_1 + \ldots + A_n$ where A_i is artinian for $1 \leq i \leq n$, then A is artinian.

The proof is similar to the proof of Proposition 1.3.

We can merge the minimal and the maximal conditions in the following way.

The module A is called a *minimax module* if it possesses a finite series of submodules with either artinian or noetherian factors.

7.2. Lemma. *Let R be an integral domain, A a finitely generated R – module. If A is R – periodic, then $Ann_R(A) \neq <0>$.*

Proof. Let M be a finite subset of A such that $A = MR$. Clearly,

$$\bigcap_{a \in M} Ann_R(a) \leq Ann_R(a).$$

Also we have

$$\Pi_{a \in M} Ann_R(a) \leq \bigcap_{a \in M} Ann_R(a).$$

Since R has no zero-divisors, $\bigcap_{a \in M} Ann_R(a) \neq <0>$. Hence $Ann_R(a) \neq <0>$.

7.3. Proposition. *Let D be a Dedekind domain, A be a periodic finitely generated D – module. Then A is a direct sum of finitely many cyclic submodules. Let*

$$A = \oplus_{\lambda \in \Lambda} a_\lambda D = \oplus_{\mu \in M} b_\mu D$$

where $Ann_D(a_\lambda D) = P_\lambda^{k_\lambda}$, $Ann_D(b_\mu D) = P_\mu^{s_\mu}$, $P_\lambda, P_\mu \in Spec(D)$, $\lambda \in \Lambda$, $\mu \in M$, and both sets Λ and M are finite. Then there are an automorphism ϕ of A and a bijection $\theta : M \longrightarrow \Lambda$ such that $b_\mu \phi = a_{\mu\theta}$ $\mu \in M$.

Proof. By Lemma 7.2 $Ann_D(a) \neq <0>$. Using Corollary 6.5 we obtain that A is a direct sum finitely many cyclic submodules. The second part follows directly from Proposition 3.8 and Theorem 6.10.

Let A be a finitely generated module over a Dedekind domain D, $T = t_D(A)$. Since D is noetherian, A is a noetherian D – module by Proposition 1.3, in particular, T is finitely generated. The factor – module A/T is D – torsion – free and finitely generated. So the next step is the consideration of torsion – free finitely generated modules.

7.4. Lemma. *Let D be a Dedekind domain, A be a finitely generated D – module. If A has an element a such that $Ann_D(a) = < 0 >$, then A includes a D – submodule B with the properties:*

(1) $A = B \oplus C$ for some D – submodule C;

(2) $a \notin B$;

(3) C is isomorphic to some ideal of D (as D – modules).

Proof. We define the mapping θ: $D \longrightarrow A$ by the rule: $x\theta = ax$, $x \in D$. Obviously, θ is a D – homomorphism. Since $Ker\ \theta = Ann_D(a) = < 0 >$, θ is a monomorphism. Let F be the field of fractions of the ring D. Since F is D – divisible, F is an injective D – module by Proposition 5.8. Therefore if ι is an identical imbedding D in F, then there is a D– homomorphism ϕ: $A \longrightarrow F$ such that $\theta\phi = \iota$. Then

$$1 = 1\ \iota = 1(\theta\phi) = (1\theta)\phi = a\phi,$$

so ϕ is a non – zero homomorphism.

Furthermore, $A\phi$ is a fractional ideal of D. It follows that $A\phi$ is isomorphic as a D – module to some non – zero ideal L of D. Since D is a Dedekind domain, L is invertible, and Proposition 4.15 implies that L is a projective D- module. From Theorem 4.12 we obtain the decomposition $A = B \oplus C$ where $B = Ker\ \phi$, $C \cong_D L$. Since $a\phi \neq 0$, $a \notin B$.

7.5. Corollary. *Let D be a Dedekind domain, A be a finitely generated D – module. If A is D – torsion – free, then $A = C_1 \oplus \ldots \oplus C_m$ where C_j is isomorphic to some ideal of D, $1 \leq j \leq m$. In particular, every torsion – free finitely generated D – module is projective.*

Proof. Let M be a maximal D-free subset of A, $E = \bigoplus_{a \in M} aD$. By Proposition 1.3, a submodule E is finitely generated. It follows that M is finite, say $M = \{ a_1, \ldots, a_m \}$. Lemma 7.4 implies that $A = C_1 \oplus B_1$ where $a_1 \notin B_1$ and C_1 is isomorphic to some ideal of D. Since A/B_1 is D – torsion – free, $a_1D \cap B_1 = < 0 >$. If M_1 is a maximal D – free subset of B_1, then $M_1 \cup \{a_1\}$ is a maximal D – free subset of A. Corollary 4.6 implies that $|M_1| = m - 1$, so $M_1 = \{ b_1, \ldots, b_{m-1} \}$. Applying Lemma 7.4 to the submodule B_1 we obtain the decomposition $B_1 = C_2 \oplus B_2$ where C_2 is isomorphic to some ideal of D, $b_1 \notin B_2$. Since A/B_2 is D – torsion – free, $(a_1D \oplus b_1D) \cap B_2 = < 0 >$. With the help of the similar arguments we finally obtain the decomposition

$$A = C_1 \oplus \ldots \oplus C_m \oplus E_1$$

where C_j is isomorphic to some ideal of D, $1 \leq j \leq m$. Since

$$r_0(C_1 \oplus \ldots \oplus C_m) = m = r_0(A),$$

$E_1 = < 0 >$. *So* $A = C_1 \oplus \ldots \oplus C_m$.

7.6. Corollary. *Let D be a Dedekind domain, A be a finitely generated D – module. If A is D – torsion – free, then*

$$A \cong \underbrace{D \oplus \ldots \oplus D}_{m-1} \oplus L$$

for some $m \geq 1$ and some non – zero ideal L of D where $m = r_0(A)$ and L is determined up to multiplication by a non – zero constant in the field of fractions of D.

This statement follows from Corollary 7.5 and Theorem 4.25.

7.7. Corollary. *Let D be a Dedekind domain, A be a finitely generated D – module. Then $A = T \oplus C$ where $T = t_D(A)$, C is a finitely generated D – torsion – free submodule.*

This statement follows from Corollary 7.5 and Theorem 4.12.

Now we will formulate the basic result.

7.8. Theorem. *Let D be a Dedekind domain, A be a finitely generated D – module. Then the following assertions hold.*

(1) $A = T \oplus C$ where T is a D – periodic part of A, C is a finitely generated D – torsion – free submodule;

(2) $T = A_1 \oplus \ldots \oplus A_k$ where A_j is a primary cyclic submodule, $1 \leq j \leq k$.

(3) If $T = B_1 \oplus \ldots B_t$ where B_j is a primary cyclic submodule, then $k = t$ and there are a permutation $\sigma \in S_k$ and an automorphism ϕ of the submodule T such that $B_j\phi = A_{j\sigma}$, $1 \leq j \leq k$.

(4) $C \cong \underbrace{D \oplus \ldots \oplus D}_{m-1} \oplus L$

for some $m \geq 1$ and some non – zero ideal L of D where $m = r_0(A)$ and L is determined up to multiplication by a non-zero constant in the field of fractions of D.

Statement (1) is Corollary 7.7. Statements (2) and (3) follow from Proposition 7.3, and statement (4) is Corollary 7.6.

7.9. Lemma. *Let D be a Dedekind domain, P a maximal idea of D, A a P – module over a ring D. If $\dim_{D/P} A/AP$ is finite, then $A = B \oplus C$ where B is a finitely generated submodule, C is a divisible submodule.*

Proof. Let

$$A/AP = \bigoplus_{1 \leq j \leq n} (a_j D + AP),\ A_0 = \sum_{1 \leq j \leq n} a_j D,\ \text{then } A = A_0 + AP$$

Since A_0 is a finitely generated P – module, there is a number s_1 such that $P^{s_1} \leq Ann_D(A_0)$. Therefore there exists a number s_2 such that $A_0 y^{s_2} = < 0 >$ but $A_0 y^{s_2 - 1} \neq < 0 >$. Put $C = Ay^{s_2}$, then

$$C = Ay^{s_2} = (A_0 + AP)y^{s_2} = (A_0 + Ay)y^{s_2} = A_0 y^{s_2} + Ay^{s_2 + 1} = Ay^{s_2 + 1} = (Ay^{s_2})y = Cy.$$

Propositions 6.1 and 6.2 imply that C is a divisible submodule. Theorem 5.13 yields that $A = C \oplus B$ for some submodule B. Furthermore

$$A = A_0 + AP = A_0 + (A_0 + AP)P =$$
$$A_0 + AP^2 = \ldots = A_0 + AP^{s(2)} = A_0 + C.$$

Now $B \cong A/C = (A_0 + C)/C \cong A_0/(A_0 \cap C)$ is a finitely generated submodule.

7.10. Corollary. *Let D be a Dedekind domain, P a maximal ideal of D, A a P – module over the ring D. If A is an artinian D – module, then*

$$A = a_1 D \oplus \ldots \oplus a_s D \oplus C_1 \oplus \ldots \oplus C_t$$

where C_j is a Prüfer P – module, $1 \leq j \leq t$.

Proof. We can consider A/AP as a vector space over the field D/P. Since A is artinian, $\dim_{D/P} A/AP$ is finite. Lemma 7.9 implies that $A = B \oplus C$ where B is a finitely generated submodule, C is a divisible submodule. Then

$$B = a_1 D \oplus \ldots \oplus a_s D$$

by Proposition 7.3. Finally, we can apply Theorem 5.26 to C.

7.11. Proposition. *Let D be a Dedekind domain, P a maximal ideal of D, A a P – module over a ring D. Then A is an artinian module if and only if $\Omega_{P,1}(A)$ is artinian.*

Proof. If A is artinian, then every submodule of A is artinian. In particular, $\Omega_{P,1}(A)$ is likewise artinian.

Conversely, suppose that $B = \Omega_{P,1}(A)$ is artinian. Let

$$A_1 \geq A_2 \geq \ldots \geq A_n \geq \ldots$$

be a descending chain of submodules of A. Choose an element $y \in P \setminus P^2$. For any pair k, $m \in \mathbf{N}$, we put

$$B_{k,m} = (A_k \cap \Omega_{\alpha, m+1}(A))y^m.$$

Obviously, each $B_{k,m}$ is a D – submodule, $B_{k,m} \geq B_{k+1,m}$ and $B_{k,m} \geq B_{k,m+1}$ for any pair $k, m \in \mathbf{N}$. It follows that $B_{k,m} \geq B_{s,t}$ whenever $s \geq k$ and $t \geq m$.

Since B is artinian, there are some $q, r \in \mathbf{N}$ such that $B_{k,m} = B_{q,r}$ whenever $k \geq q$ and $m \geq r$. The mapping $\phi : a \longmapsto a\,y^l$, $a \in \Omega_{P,t-1}(A)$, is a module endomorphism, and so $Im\ \phi$ and $Ker\ \phi = \Omega_{P,1}(A)$ are submodules. Therefore

$$\Omega_{P,t+1}(A)/\Omega_{P,t}(A) \cong Im\ \phi \leq \Omega_{P,1}(A).$$

It follows that $\Omega_{P,t+1}(A)/\Omega_{P,t}(A)$ is artinian and therefore so $\Omega_{P,n}(A)$ does for any $n \in \mathbf{N}$.

Therefore there is some $q(1) \in \mathbf{N}$ such that

$$A_k \cap \Omega_{P,r+1}(A) = A_{q(1)} \cap \Omega_{P,r+1}(A)$$

for every $k \geq q(1)$. Let $q(2)$ be the maximum of $\{q, q(1)\}$. If $k \geq q(2)$ and $n \geq r+1$, then we claim that the following expression holds:

(E) $$A_k \cap \Omega_{P,n}(A) = A_{q(2)} \cap \Omega_{P,n}(A).$$

To see this, we proceed by induction on n. Suppose that (E) has been shown for some $n \geq r+1$. Pick $a \in A_{q(2)} \cap \Omega_{P,n+1}(A)$. Then $a\,y^n \in B_{q(2),n} = B_{k,n}$ so that $a\,y^n = y\,\alpha^n$ for some $c \in A_k \cap \Omega_{P,n+1}(A)$. It follows that $(a-c)y^n = 0$, that is,

$$a - c \in A_{q(2)} \cap \Omega_{P,n}(A) = A_k \cap \Omega_{P,n}(A).$$

In particular, $a \in A_k$, that is, $a \in A_k \cap \Omega_{P,n+1}(A)$. Consequently,

$$A_k \cap \Omega_{P,n+1}(A) = A_{q(2)} \cap \Omega_{P,n+1}(A),$$

and then our claim follows. Hence

$$A_k = A_k \cap A = A_k \cap (\bigcup_{n \geq r+1} \Omega_{P,n}(A)) =$$
$$\bigcup_{n \geq r+1} (A_k \cap \Omega_{P,n}(A)) = \bigcup_{n \geq r+1} (A_{q(2)} \cap \Omega_{P,n}(A)) =$$
$$A_{q(2)} \cap (\bigcup_{n \geq r+1} \Omega_{P,n}(A)) = A_{q(2)},$$

and so A is artinian.

7.12. Corollary. *Let D be a Dedekind domain, P a maximal ideal of D, A a P – module over a ring D. Then A is an artinian module if and only if $dim_{D/P}\ \Omega_{P,1}(A)$ is finite.*

Proof. If A is artinian, then $\Omega_{P,1}(A)$ is also artinian. It follows that $dim_{D/P}\ \Omega_{P,1}(A)$ is finite. Proposition 7.11 implies the converse.

7.13. Theorem. *Let D be a Dedekind domain. Then a D – module A is artinian if and only if*

$$A = a_1D \oplus \ldots \oplus a_nD \oplus C_1 \oplus \ldots \oplus C_k$$

where C_i *is a Prüfer* P_i *– module,* $P_i \in Spec(D)$, $1 \leq i \leq k$, $Ann_D(a_j) = P_j^{m_j}$, $P_j \in Spec(D)$, $1 \leq j \leq n$.

Proof. Suppose that A is artinian. If A is not D – periodic, then A has an element a with the property $Ann_D(a) = < 0 >$. Therefore $aD \cong D$. In Theorem 1.29 we have already proved that D has an infinite descending chain

$$D > P > P^2 > \ldots > P^n > \ldots$$

for every $P \in Spec(D)$. This proves that A is D – periodic. Corollary 3.8 implies that $A = \bigoplus_{P \in \pi} A_P$, where $\pi = Ass_D(A)$, A_P is the P – component of A. Since A is artinian, the set $Ass_D(A)$ is finite. Now we can apply Corollary 7.10 to each submodule A_P. The sufficiency follows from Corollary 7.12 and Lemma 7.1.

Let R be a noetherian ring, A a minimax R – module. Consider a finite series

$$< 0 > = A_0 \leq A_1 \leq \ldots \leq A_n = A$$

of R – submodules of A with artinian or noetherian factors. If Δ is the set of indexes j such that A_j/A_{j-1} is noetherian, for every $j \in \Delta$ then there exists a finite subset $M_j \subseteq A$ such that $A_j = M_jR + A_{j-1}$. *If* $M = \bigcup_{j \in \Delta} M_j$, then M is finite, $B = MR$ is noetherian and A/B is artinian. Hence A is noetherian – by – artinian.

Furthermore, suppose that $R = D$ is a Dedekind domain, which is not a field, and take P to be a non – zero maximal ideal of D. In Theorem 1.29 we have already proved that $P^n \neq P^{n-1}$ and then every cyclic torsion – free D – module cannot be artinian. Thus A/B has to be D – periodic and, since B is finitely generated by Theorem 7.8. We may decompose $B = T \oplus C$, where T is the periodic part of B and C is a finitely generated projective submodule of B. By Lemma 7.1 A/C is periodic artinian. Thus in this case the minimax D-module includes a finitely generated projective D – submodule C such that A/C is artinian.

If D is a field, we simply note that *a minimax* D *– module has finite dimension* (as a D – space).

Chapter



Pure Submodules

In the current chapter we extend the important concept of pure subgroups on modules over Dedekind domains. This concept plays a significant role in the theory of abelian groups and very useful in periodic and torsion free abelian groups.

Let D be a Dedekind domain, A a $D-$module, $x \in D$. A submodule B of A is said to be $x-pure$ if $Bx = B \cap Ax$.

If L is an ideal of D and B is an $x-$pure submodule of A for every $x \in L$, then B is called an $L-pure\ submodule$. B is called a *pure submodule* if it is $D-$pure.

8.1. Lemma. *Let D be a Dedekind domain, $x \in D$, A a $D-$module. If $\mathcal{S}$ is a linearly ordered (by inclusion) set of $x-$pure submodules, then the submodule $B = \bigcup \mathcal{S}$ is also $x-$pure.*

Proof. Let $b \in Ax \cap B$. There is $C \in \mathcal{S}$ such that $b \in C$, that is $b \in Ax \cap C = Cx$, so that $b \in Cx$. The inclusion $Bx \leq Ax \cap B$ is obvious.

8.2. Lemma. *Let D be a Dedekind domain, $P \in Spec\ (D)$, A a D – module, B a P – submodule of A. Then B is D – pure if and only if B is P – pure.*

Proof. If B is a D – pure submodule, then B is also P – pure. Conversely, let B be P – pure, $x \in D$. If $x \notin P$, then $Bx = Ax \cap B$. Let $x \in P$, then $xD + P = D$. If $a \in Ax \cap B$, then $Ann_D(a) = P^k$ for some $k \in \mathbb{N}$. Lemma 1.18 implies that $xD + P^k = D$, that is $1 = x u + v$, $u \in D$, $v \in P^k$. Now we have

$$a = a\,1 = a(x\,u + v) = a(u\,x) + a\,v = (au)x.$$

Since B is a submodule, $au \in B$, so that $a \in Bx$. The inclusion $Bx \leq Ax \cap B$ is obvious.

8.3. Lemma. *Let D be a Dedekind domain, $x \in D$, A a D – module. If $A = B \oplus C$, then the submodules B and C are x – pure.*

Proof. Let $b \in B \cap Ax$, then $b = ax$ for some element $a \in A$. Since $a = u + v$, $u \in B$, $v \in C$, $ax = ux + vx = b$. It follows that $vx = b - ux \in B$, that is $vx = 0$ and $b = ux$. So $B \cap Ax \leq Bx$.

8.4. Lemma. *Let D be a Dedekind domain, $x \in D$, A a D – module, B and C be submodules of A, $C \leq B$.*

(1) If B, C are x – pure submodules of A, then C is an x – pure submodule of B.

(2) If C is an x – pure submodule of B, B is an x – pure submodule of A then C is an x – pure submodule of A.

(3) If B is an x – pure submodule of A, then B/C is an x – pure submodule of A/C.

(4) If C is an x – pure submodule of A, B/C is an x – pure submodule of A/C, then B is an x – pure submodule of A.

Proof. (1). We have $Cx = Ax \cap C = Ax \cap (C \cap B) = (Ax \cap B) \cap C = Bx \cap C$.
(2) Follows from the equations

$$Cx = C \cap Bx = C \cap (B \cap Ax) = (C \cap B) \cap Ax = C \cap Ax.$$

(3) For every submodule $E \geq C$ we have $(E/C)x = Ex + C/C$. Therefore $(B/C) \cap (A/C)x = (B/C) \cap (Ax + C/C)$. Let $b + C \in (B/C) \cap (A/C)x$ where $b \in B$. Then $b + C = ax + C$ for some element $a \in A$; that is $b = ax + c$ where $c \in C$. It follows that $b - c = u = ax$. Since $C \leq B$, $u \in B$, that is, $u \in B \cap Ax = Bx$. But $u + C = b + C$, so $b + C \in (Bx + C)/C = (B/C)x$.

(4). Let $b \in Ax \cap B$, i.e. $b = ax$ for some $a \in A$. Then

$$b + C \in (Ax + C)/C = (A/C)x,$$

that is, $b + C \in (A/C)x \cap (B/C)$. Since B/C is a pure submodule of A/C,

$$(A/C)x \cap (B/C) = (B/C)x = (Bx + C)/C,$$

so $b + C = ux + C$ for some element $u \in B$. Then $ux = b + c$, $c \in C$, so $ux = ax + c$ and $(u - a)x = c$. Since $C \cap Ax = Cx$, $(u - a)x = vx$ for some element $v \in C$. So $ax = ux - vx = (u - v)x \in Bx$. This means that $Ax \cap B = Bx$.

8.5. Lemma. *Let D be a Dedekind domain, $x \in D$, A a D – module, B an x – pure submodule.*

(1) The D – periodic part of B is an x – pure submodule of A.

(2) The P – component of B is an x – pure submodule of A for every $P \in Spec(D)$.

Proof. Let $T = t_D(A)$, $b \in T \cap Ax$. Then $b \in B \cap Ax = Bx$, so there is an element $u \in B$ such that $b = ux$. We can assume that $x \neq 0$. If $0 \neq z \in Ann_D(b)$, then $0 = bz = (ux)z = u(xz)$, that is $0 \neq xz \in Ann_D(u)$. So $u \in T$, i.e. $T \cap Ax = Tx$.

(2) By (1), taking in account Lemma 8.4 we can assume that B is a D– periodic submodule. Corollary 3.8 implies that $B = \bigoplus_{P \in \pi} B_P$ where B_P is a P – component of B, $\pi = Ass_D(B)$. Let $b \in B_P \cap Ax$, $P \in \pi$ Then $b \in B \cap Ax = Bx$, i.e. $b = wx$ for some element $w \in B$.

We have $w = \sum_{P \in \sigma} e_P$ where σ is a finite subset of π, $e_P \in B_P$. Then

$$wx = (\sum_{P \in \sigma} e_P)x = \sum_{P \in \sigma} e_P x.$$

Since $wx - b \in B_P$, then $\sum_{Q \in \sigma, Q \neq P} e_P x = 0$. So $b = wx = (e_P)x$ and $e_P \in B_P$. Thus $b \in B_P x$.

The following assertion illustrates the role of pure submodules in P – modules.

8.6. Proposition. *Let D be a Dedekind domain, $P \in Spec(D)$, A a D – module, B a P – submodule of A, C a pure submodule of B. If $\Omega_{P,1}(B) = \Omega_{P,1}(C)$, then $B = C$.*

Proof. We will prove that $\Omega_{P,k}(B) = \Omega_{P,k}(C)$ for every $k \in \mathbb{N}$. It follows that $B = C$. We apply the induction on k. For $k = 1$ all is obvious. Let $k > 1$, and let we have already proved that $\Omega_{P,k-1}(B) = \Omega_{P,k-1}(C)$. Let $b \in \Omega_{P,k}(B)$, $y \in P \setminus P^2$. Proposition 1.24 shows that $P = yD + P^k$. Since

$$P^k = Ann_D(\Omega_{P,k}(B)), \quad (\Omega_{P,k}(B))P = (\Omega_{P,k}(B))y,$$

so

$$by \in \Omega_{P,k-1}(B) = \Omega_{P,k-1}(C).$$

Thus $by \in C \cap By = Cy$, that is, $by = cy$ for some element $c \in C$. It follows that $(b - c)y = 0$, that is $b - c \in \Omega_{P,1}(B) = \Omega_{P,1}(C)$. This means that $b \in C$, so $\Omega_{P,k}(B) = \Omega_{P,k}(C)$.

8.7. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, $y \in P \setminus P^2$, A a D – module, B a P – submodule of A, C a maximal submodule of A with the following property: $C \cap B = < 0 >$. If a is an element such that $ay \in C$, then $a \in C \oplus B$.*

Proof. Suppose that A/C is not periodic. Then there is an element $e \in A$ such that $Ann_D(e + C) = < 0 >$. Hence

$$(e + C)D \cong D \text{ and } (e + C)D \cap (B + C)/C = < 0 >,$$

in particular, $(eD + C) \cap B = < 0 >$. However, this contradicts the choice of C. Consequently, A/C is a D – periodic module. Corollary 3.8 implies that $A/C = \oplus_{Q \in \pi} (A/C)_Q$ where $(A/C)_Q$ is a Q – component of B, $\pi = Ass_D(A/C)$. Put $E/C = \oplus_{Q \neq P} (A/C)_Q$. Since

$$(B + C)/C \leq (A/C)_P, (B + C/C) \cap (E/C) = < 0 >.$$

It follows that $B \cap E = < 0 >$. The inclusion $C \leq E$ implies that $C = E$, in particular, A/C is a P – module. Hence there exists a number $k \in \mathbb{N}$ such that $aP^k \leq C$. By the choice of C we obtain that $B \cap (C + aD) \neq < 0 >$, so there is an element $0 \neq b = c + ax$ where $c \in C$, $x \in D$. Suppose that $x \in P$. Proposition 1.24 shows that $P = yD + P^k$, so $x = yu + v$ where $u \in D$, $v \in P^k$. So we have $ax = ayu + av \in C + aP^k \leq C$. In this case $b = c + ax \in C$, that is $b \in B \cap C = < 0 >$. This contradiction shows that $x \notin P$, thus $xD + P = D$. Lemma 1.18 implies that $xD + P^k = D$. Therefore, $1 = xw + z$ where $w \in D$, $z \in P^k$. We have now

$$a = a1 = axw + az = (b - c)w + az \in B + C.$$

8.8. Lemma. *Let D be a Dedekind domain, L an ideal of D, A a D – module, T the D – periodic part of A. If $TL = < 0 >$, then $AL \cap T = < 0 >$.*

Proof. Assume the contrary. Let $0 \neq a \in AL \cap T$. Then $a = a_1 x_1 + \ldots + a_n x_n$ for some elements $a_1, \ldots, a_n \in A$, $x_1, \ldots, x_n \in I$. Put $B = a_1 D + \ldots + a_n D$. Corollary 7.7 yields that $B = T_1 \oplus C$ where $T_1 = T \cap B$, C is a D – torsion – free submodule. For every element $b \in B$ we have $b = u + c$, where $u \in T_1$, $c \in C$. If $z \in L$, then $bz = uz + cz = cz$. It follows that $BL \leq C$, in particular, $BL \leq T_1 = < 0 >$. But $a = a_1 x_1 + \ldots + a_n x_n \in BL$ and $a \in B \cap T = T_1$, so we have contradiction, which shows that $AL \cap T = < 0 >$.

8.9. Proposition. *Let D be a Dedekind domain, A a D – module, $T = t_D(A)$. If $Ann_D(T) \neq < 0 >$, then $A = T \oplus C$ for some submodule C.*

Proof. Let $L = Ann_D(T)$. Theorem 1.14 yields that $L = P_1^{k_1} \ldots P_t^{k_t}$ where $P_i \neq P_j$ for $i \neq j$, $k_1, \ldots, k_t \in \mathbf{N}$, $P_1, \ldots, P_t \in Spec(D)$. We will prove this Proposition using the induction on $k(L) = k_1 + \ldots + k_t$. If $k(L) = 1$, then $L \in Spec(D)$. By Lemma 8.8 $AL \cap T = < 0 >$. We can consider A/AL as a vector space over the field D/L. Since every subspace of vector space has a complement, there is a submodule $C \geq AL$ such that $A/AL = (T + AL/AL) \oplus (C/AL)$. It follows that $A = C + T$ and $C \cap T \leq AL$, that is, $T \cap C = < 0 >$. In other words $A = C \oplus T$.

Suppose now that $k(L) > 1$. Let $P \in \{P_1, \ldots P_t\}$, U be the P – component of T, $V = \Omega_{P, 1}(U)$. Then $k(Ann_D(T/V)) = k(L) - 1$. By the induction hypothesis $A/V = T/V \oplus B/V$ for some submodule B. We have $V = t_D(B)$ and $Ann_D(V) = P$. We have already proved that $B = V \oplus C$ for some submodule C. Moreover, C is D – torsion – free, so $C \cap T = < 0 >$. Thus $A = T + B = T + V + C = T + C$, that is, $A = T \oplus C$.

8.10. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A a D – module, B a submodule A such that $B = \oplus_{\lambda \in \Lambda} a_\lambda D$ where $Ann_D(a_\lambda) = P^k$ for every $\lambda \in \Lambda$. If B is a pure submodule of A, then $A = B \oplus C$ for some submodule C.*

Proof. Firstly assume that A is a P - module. Let $b \in B \cap AP^k$. Then $b = a_1 x_1 + \ldots + a_n x_n$ for some elements $a_1, \ldots, a_n \in A$, $x_1, \ldots, x_n \in P^k$. Let $Ann_D(a_j) = P^{q_j}$, $1 \leq j \leq n$. Put $m = max\{k, q_1, \ldots, q_n\}$. Let $y \in P \setminus P^2$. Proposition 1.24 yields that $P^k = y^k D + P^m$, so that $x_j = y^k z_j + u_j$ where $z_j \in D$, $u_j \in P^m$, $1 \leq j \leq n$. We have

$$b = a_1 y^k z_1 + a_1 u_1 + \ldots + a_n y^k z_n + a_n u_n = (a_1 z_1 + \ldots + a_n z_n) y^k = e y^k.$$

From the equation $B \cap Ay^k = By^k$ it follows that $b = ey^k = b_1 y^k$ for some $b_1 \in B$. However, $b_1 y^k = 0$. Hence $AP^k \cap B = < 0 >$.

Let C be a maximal submodule of A with the properties: $AP^k \leq C$ and $B \cap C = < 0 >$. Suppose that $A \neq B \oplus C$. Let $B \oplus C \neq v + (B \oplus C) \in \Omega_{P, 1}(A/(B \oplus C))$. Proposition 1.24 implies that $yD + P^k = P$. By the selection of v we obtain that $vy \in B \oplus C$, that is $vy = b_2 + c$ where $b_2 \in B$, $c \in C$. Therefore

$$v y^k = (vy) y^{k-1} = b_2 y^{k-1} + c y^{k-1},$$

so $b_2 y^{k-1} = v y^k - c y^{k-1} \in C$, because $Ay^k \leq C$. This means that $b_2 y^{k-1} \in C \cap B = < 0 >$, that is, $b_2 y^{k-1} = 0$. Thus

$$b_2 \in \Omega_{P, k-1}(B) = \oplus_{\lambda \in \Lambda} a_\lambda P = \oplus_{\lambda \in \Lambda} a_\lambda y D,$$

so $b_2 = \sum_{\lambda \in \Delta} a_\lambda y s_\lambda$ where $s_\lambda \in D$, Δ is a finite subset of Λ. Put $b_3 = \sum_{\lambda \in \Delta} a_\lambda s_\lambda$ then $b_2 = b_3 y$. Thus we have

$$vy = b_2 + c = b_3 y + c \text{ and } c = vy - b_3 y = (v - b_3) y.$$

If we put $v_1 = v - b_3$, then $v_1 \notin B \oplus C$ and $v_1 y \in C$. Lemma 8.7 yields that $v_1 \in B \oplus C$. This contradiction shows that $A = B \oplus C$.

Assume now that A is D – periodic. Corollary 3.8 implies that $A = \oplus_{Q \in \pi} A_Q$ where $\pi = Ass_D(A)$, A_Q is the Q – component of A. Put $E = \oplus_{Q \neq P} A_Q$, then $A = A_P \oplus E$. By Lemma 8.3 A_P is a pure submodule of A, and Lemma 8.4 yields that B is a pure submodule of A_P. We have already proved that $A_P = B \oplus C_1$ for some submodule C_1. Then $A = B \oplus C$ where $C = E \oplus C_1$.

Finally consider a general case. Let $T = t_D(A)$. Lemmas 8.3 and 8.4 prove that B is a pure submodule of T. We have already proved that $T = B \oplus C_2$ for some submodule C_2. By $T/C_2 = (B \oplus C_2)/C_2 \cong B/(B \cap C_2) \cong B$ we obtain $Ann_D(T/C_2) = P^k$. By Proposition 8.9, there exists a submodule $C \geq C_2$ such that $A/C_2 = (T/C_2) \oplus (C/C_2)$. Then

$$A = T + C = (B \oplus C_2) + C = B + C$$

and $B \cap C = < 0 >$, so that $A = B \oplus C$.

8.11. Theorem. *Let D be a Dedekind domain, A a D – module, B a pure submodule of A. If $Ann_D(B) = L \neq < 0 >$, then $A = B \oplus C$ for some submodule C.*

Proof. Theorem 1.14 yields that $L = P_1^{k_1} \ldots P_t^{k_t}$ where $P_i \neq P_j$ for $i \neq j$, $k_1, \ldots, k_t \in \mathbb{N}$, $P_1, \ldots, P_t \in Spec(D)$. We will use the induction on $k(L) = k_1 + \ldots + k_t$. If $k(L) = 1$, then $L \in Spec(D)$. In this case, B is a vector space over the field D/L, therefore $B = \oplus_{\lambda \in \Lambda} a_\lambda D$ where $Ann_D(a_\lambda) = L$ for any $\lambda \in \Lambda$. Lemma 8.10 implies that $A = B \oplus C$ for some submodule C.

Suppose now that $k(L) > 1$. Corollary 6.5 yields that B is the direct sum of cyclic submodules. Taking into account Corollary 3.8, we obtain that there is a number $m \leq k_1$ such that $B = B_1 \oplus B_2$ where $B_1 = \oplus_{\mu \in M} b_\mu D$ and $Ann_D(b_\mu) = P_1^m$ for any $\mu \in M$, and $Ann_D(B_2) \leq P_2^{k_2} \ldots P_t^{k_t}$. In particular, $k(Ann_D(B_2)) < k(L)$. It follows from Lemma 8.3 that B_1 is a pure submodule of A. By Lemma 8.10 there exists a submodule C_1 such that $A = B_1 \oplus C_1$. Since $B_1 \leq B$, $B = B_1 \oplus (C_1 \cap B)$. Put $B_3 = C_1 \cap B$; then $B_3 \cong B/B_1 \cong B_2$, in particular, $k(Ann_D(B_3)) < k(L)$. Let $b \in B_3 \cap C_1 x$, $x \in D$. Then $b \in B \cap Ax = Bx$, that is, $b = b_1 x$ for some elements $b_1 \in B$. We have $b_1 = b_2 + b_3$, $b_2 \in B_1$, $b_3 \in B_3$. Therefore $b = b_1 x = b_2 x + b_3 x$. It follows that $b_2 x = b - b_3 x \in B_3$, so $b_2 x = 0$ thus $b = b_3 x$, i.e. $B_3 \cap C_1 x \leq B_3 x$.

This means that B_3 is a pure submodule of C_1. By the induction hypothesis $C_1 = B_3 \oplus C$ for some submodule C. Hence

$$A = B_1 \oplus C_1 = B_1 \oplus B_3 \oplus C = B \oplus C.$$

Theorem 8.11 is an analog of the famous Prüfer – Kulikov Theorem (see, [FL1970, Theorem 27.5]).

8.12. Proposition. *Let D be a Dedekind domain, $P \in Spec(D)$, A a P – module over the ring D, $k \in \mathbb{N}$. If B is the maximal submodule of A with the property*

$B \cap AP^k = < 0 >$, *then* B *is a pure submodule.*

Proof. Since $BP^k \leq AP^k$ and

$$BP^k \leq B,\ BP^k \leq AP^k \cap B = < 0 >.$$

Let $y \in P \setminus P^2$. We will prove that $Ay^r \cap B \leq By^r$ for every $r \in \mathbf{N}$. We will use the induction on r. For $r = 0$ we have $Ay^0 \cap B = A \cap B = B = By^0$. Let $r > 1$. Assume that $Ay^r \cap B = By^r$. Let $0 \neq b \in Ay^{r+1} \cap B$, that is $b = a\,y^{r+1}$ for some $a \in A$. Then $(ay^r)y \in B$, and Lemma 8.7 yields that $ay^r \in B \oplus AP^k$. If we assume that $r + 1 > k$, then

$$a\,y^{r+1} = ay^{k+(r+1-k)} = a\,y^{r+1-k}\,y^k \in Ay^r,$$

so $b \in AP^k \cap B = < 0 >$. This contradiction shows that $r + 1 \leq k$, i.e. $r \leq k - 1$. We have $ay^r = b_1 + c$ where $b_1 \in B$, $c \in AP^k$. In other words, $c = a_1x_1 + \ldots + a_nx_n$ for some $a_1, \ldots, a_n \in A$, $x_1, \ldots, x_n \in P^k$. Since A is a P – module, $Ann_D(a_j) = P^{m_j}$, $1 \leq j \leq n$. Put $m = max\{m_1, \ldots, m_n\}$. Proposition 1.24 implies that $P^k = y^kD + P^m$, so that $x_j = y^kz_j + w_j$ where $w_j \in P^m$, $z_j \in D$, $1 \leq j \leq n$. Then

$$a\,y^r = b_1 + c = b_1 + \sum_{1 \leq j \leq n} (a_j\,y^kz_j + a_j\,w_j) = b_1 + \sum_{1 \leq j \leq n} a_j\,y^kz_j = b_1 + c_1y^k$$

where $c_1 = \sum_{1 \leq j \leq n} a_j\,z_j$. It follows that

$$b_1 = ay^r - c_1y^k = (a - c_1y^{k-r})\,y^r,$$

thus $b_1 \in Ay^r \cap B = By^r$. Hence there is an element $b_2 \in B$ such that $b_1 = (b_2)y^r$. In this case

$$b = a\,y^{r+1} = (a\,y^r)y = (b_1 + c)y = b_1y + cy.$$

It follows that $cy = b - b_1y \in B \cap AP^k = < 0 >$, so $b = b_1y = b_2y^{r+1}$. This means that $Ay^{r+1} \cap B = By^{r+1}$.

Let $u \in P$, $0 \neq b_3 \in Au \cap B$, that is $b_3 = eu$ for some $e \in A$. There is a number $r \in \mathbf{N}$ such that $u \in P^r \setminus P^{r+1}$, because $\bigcap_{r \in \mathbf{N}} P^r = < 0 >$. Put $P^s = Ann_D(e)$. Since $b_3 \neq 0$, $r \leq s$ and we can assume that $s > k$. Proposition 1.24 yields that $P^r = y^rD + P^s$, therefore $u = y^ru_1 + u_2$, $u_1 \in D$, $u_2 \in P^s$. We have $b_3 = e\,y^ru_1 + e\,u_2 = e\,u_1y^r$, that is, $b_3 \in Ay^r \cap B = By^r$ and $b_3 = b_4y^r$. Since $u \notin P^{r+1}$, $u \notin P$. Hence $u_1D + P + D$. Lemma 1.18 implies that $u_1D + P^k = D$, thus $1 = u_1\,v_1 + v_2$ where $v_1 \in D$, $v_2 \in P^k$. Now we obtain the equations

$$b_4 = b_4\,v_1\,u_1 + b_4\,v_2 = (b_4\,v_1)\,u_1 = b_5\,u_1$$

where $b_5 = b_4v_1 \in B$. Hence $b_3 = b_4\,y^r = b_4\,u_1y^r$. Since $s > k$, $b_5\,u_2 = 0$, and, finally,

$$b_3 = b_5 u_1 y^r + b_5 (u_2) = b_5 (u_1 y^r + u_2) = b_5 u.$$

Consequently, $Au \cap B = Bu$. Now we can use Lemma 8.2.

Finally, we present some results on pure submodules in torsion – free modules.

8.13. Proposition. *Let D be a Dedekind domain, A a D – torsion – free D – module, B a submodule of A. Then B is pure if and only if A/B is D – torsion – free.*

Proof. Let B be a pure submodule of A, $T/B = t_D(A/B)$, $a \in T$. Then $ax \in B$ for some element $0 \neq x \in D$. Since $B \cap Ax = Bx$, $ax = bx$ for some element $b \in B$. It follows that $(a - b)x = 0$, that is, $x \in Ann_D(a - b)$. Since A is D – torsion – free, $a - b = 0$, that is, $a = b$. This means that $T = B$, in other words A/B is D – torsion – free.

Conversely, let A/B be D – torsion – free, $ax \in B$ for some $x \in D$. If $a \notin B$, then $a + B$ is a non – zero D – periodic element of A/B. But this is impossible. Hence $a \in B$, that is, $Ax \cap B = Bx$ and B is a pure submodule.

8.14. Lemma. *Let D be a Dedekind domain, $x \in D$, A a D – torsion – free D – module, $\mathfrak{B}$ a family of x – pure submodules, $B = \bigcap \mathfrak{B}$. Then B is an x – pure submodule.*

Proof. Let $ax \in B \cap Ax$, then $ax \in M \cap Ax$ for every submodule $M \in \mathfrak{B}$. Since M is x – pure, $M \cap Ax = Mx$, so $ax = (c_M)x$ for some elements $c_M \in M$. It follows that $(a - c_M)x = 0$. Since A is torsion – free, $a - c_M = 0$, thus $a = c_M$ and $a \in M$ for every $M \in \mathfrak{B}$. Thus $a \in \bigcap \mathfrak{B} = B$.

8.15. Corollary. *Let D be a Dedekind domain, $x \in D$, A a torsion – free D – module, B a submodule of A. Then there exists the least x – pure submodule C with the property $C \geq B$.*

In fact, $C = \bigcap \{ M \mid M$ is an x – pure submodule such that $B \geq M \}$.

This least x – pure submodule is called the *x – pure envelope* of B.

Similarly

8.16. Corollary. *Let D be a Dedekind domain, L an ideal of D, A a torsion – free D – module, B a submodule of A. Then there exists the least L – pure submodule C including B.*

The least L – pure submodule, including B, is called the *L – pure envelope of B*. If $L = D$, then we will say about the *pure envelope of B*.

8.17. Proposition. *Let D be a Dedekind domain, A a torsion – free D – module. Then A has an ascending series of pure submodules*

$$<0> = A_0 \leq A_1 \leq \ldots A_\alpha \leq A_{\alpha+1} \leq \ldots A_\gamma = A$$

such that $r_0(A_{\alpha+1}/A_\alpha) = 1$ *for each* $\alpha < \gamma$.

Proof. Let $0 \neq a_1 \in A$, $C_1 = a_1D$, A_1/C_1 be a D – periodic part of A/C_1. Then A/A_1 is torsion – free, hence A_1 is a pure submodule by Proposition 8.13. Obviously, $r_0(A_1) = 1$.

Let $\alpha > 1$ and suppose that we have already constructed the submodules A_β for all $\beta < \alpha$. If α is a limit ordinal, then put $A_\alpha = \cup_{\beta<\alpha} A_\beta$. Lemma 8.1 yields that A_α is a pure submodule.

Suppose now that $\alpha - 1$ exist, and

$$A_{\alpha-1} \neq a_\alpha + A_{\alpha-1}, C_\alpha/A_{\alpha-1} = (a_\alpha D + A_{\alpha-1})/A_{\alpha-1},$$

$A_\alpha/A_{\alpha-1}$ be a D – periodic part of $A/A_{\alpha-1}$. Proposition 8.13 implies that A_α is a pure submodule. Obviously, $r_0(A_\alpha/A_{\alpha-1}) = 1$.

Chapter



Basic Submodules

The detailed consideration of modules over Dedekind domains will be continued in this chapter. We will concern about module analogies of basis and p-basis subgroups of abelian groups. These subgroups have to be proven very useful in numerous group theoretical investigations.

The following question appears in connection with the main results of Chapter 6.

Does every P – module over a Dedekind domain decompose into a direct sum of cyclic submodules?

The negative answer follows from the example below.

9.1. Example. Let D be a Dedekind domain, $P \in Spec(D), A_n = D/P^n$, $n \in \mathbb{N}$, $A = \prod_{n \in \mathbb{N}} A_n$, $T = t_D(A)$. Then $|T| = 2^{|D|}$ and $|\Omega_{P,1}(T)| = 2^{|D|}$. Suppose that

$T = \oplus_{\lambda \in \Lambda} a_\lambda D$ for some elements $a_\lambda \in T$, $\lambda \in \Lambda$. Put

$$\Lambda_k = \{ \lambda \in \Lambda \mid Ann_D(a_\lambda) = P^k \}, \quad B_k = \oplus_{\lambda \in \Lambda_k} a_\lambda D, \; C_k = \Omega_{P,1}(B_k), \; k \in \mathbf{N}.$$

Then $T = \oplus_{k \in \mathbf{N}} B_k$ and $\Omega_{P,1}(T) = \oplus_{k \in \mathbf{N}} C_k$. Since $| \Omega_{P,1}(T) | = 2^{|D|}$, there is a number t such that $|C_t| = 2^{|D|}$. Put

$$C = \oplus_{k \in \mathbf{N}} C_k, \; E_n = \oplus_{m > n} C_m, \; n \in \mathbf{N}.$$

If $a \in E_n$, then $a \in TP^n$. On the other hand, $a = (a_n)_{n \in \mathbf{N}}$ where $a_n \in A_n$, $n \in \mathbf{N}$. If $a \in TP^n$, this implies that $a_1 = a_2 = \ldots = a_n = 0$. Hence, $a \in E_n$ if and only if $a_1 = a_2 = \ldots = a_n = 0$. This means that $E_n = \Omega_{P,1}(t_D(\prod_{m > n} A_m))$. In this case, $\dim_{D/P}(C/E_n) = n$. On the other hand, from the equations

$$C = C_1 \oplus C_2 \oplus \ldots \oplus C_t \oplus E_t$$

and

$$|C_t| = 2^{|D|}$$

we obtain that $\dim_{D/P}(C/E_t)$ is infinite. This contradiction shows that T cannot to be decomposed in a direct sum of cyclic submodules.

Consequently, some P – modules have a very sophisticated structure. Nevertheless, direct sums of cyclic submodules and divisible modules play a significant role here.

9.2. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, $y \in P \setminus P^2$, A a P – module over the ring D, B a submodule of A. If $\Omega_{P,1}(B) \cap Ay^k = \Omega_{P,1}(By^k)$ for every $k \in \mathbf{N}$, then B is a pure submodule of A.*

Proof. First we will prove that $\Omega_{P,n}(B) \cap Ay^k = \Omega_{P,n}(By^k)$ for every $n \in \mathbf{N}$. For $n = 1$ this assertion is true. Let $n > 1$ and we have already proved the equation $\Omega_{P,n}(B) \cap Ay^k = \Omega_{P,n}(By^k)$. Choose $b \in \Omega_{P,n+1}(B) \cap Ay^k$, so $b = ay^k$ for some element $a \in A$. Then $by = ay^{k+1}$. Since $by \in \Omega_{P,n}(B) \cap Ay^{k+1}$, from the induction hypothesis we obtain that $by = b_1 y^{k+1}$ for some $b_1 \in B$. It follows that

$$0 = b_1 y^{k+1} - by = (b_1 y^k - b)y,$$

that is,

$$b_1 y^k - b \in \Omega_{P,1}(B) \cap Ay^k = \Omega_{P,1}(By^k).$$

Hence there exists an element $b_2 \in B$ with the property $b_1 y^k - b = b_2 y^k$. It follows that

$$b = b_1y^k - b_2y^k = (b_1 - b_2)y^k \in By^k.$$

We have now

$$B \cap Ay^k = (\cup_{n \in N} \Omega_{P,n}(B)) \cap Ay^k =$$
$$\cup_{n \in N} (\Omega_{P,n}(B)) \cap Ay^k) = \cup_{n \in N} \Omega_{P,n}(By^k) = By^k.$$

As in the second part of the proof of Proposition 8.12, we can show that B is a P – pure submodule. Lemma 8.2 implies that B is a pure a submodule of A.

9.3. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, $y \in P \setminus P^2$, A a P – module over the ring D. If $\Omega_{P,1}(A) \leq A(y^k)$ for every $k \in \mathbb{N}$, then A is a divisible module.*

Proof. We will use induction on n to prove the inclusion $\Omega_{P,n}(A) \leq Ay^k$ for every $k \in \mathbb{N}$. For $n = 1$ this is true. Suppose that $n > 1$ and that we have already proved the inclusion $\Omega_{P,n}(A) \leq Ay^k$ for every $k \in \mathbb{N}$. Let $a \in \Omega_{P,n+1}(A)$. Then $ay^n \in \Omega_{P,1}(A)$. Since $\Omega_{P,1}(A) \leq Ay^{k+n}$, $ay^n = by^{k+n}$ for some element $b \in A$. It follows that $0 = (a - by^k)y^n$, that is, $a - by^k \in \Omega_{P,n}(A)$. Since $\Omega_{P,n}(A) \leq Ay^k$, $a - by^k = (b_1)\, y^k$ for some element $b_1 \in A$. So

$$a = by^k - b_1y^k = (b - b_1)y^k \in Ay^k.$$

In particular, $A = \cup_{n \in N} \Omega_{P,n}(A) \leq Ay$. Proposition 6.1 yields that A is a divisible module.

Let D be a Dedekind domain, $P \in Spec(D)$, A a P – module over the ring D. A submodule B is said to be a *basic submodule* if it satisfies the following conditions:

(1) B is a direct sum of cyclic submodules;
(2) B is a pure submodule of A;
(3) A/B is a divisible module.

9.4. Theorem *Let D be a Dedekind domain, $P \in Spec(D)$. Every P-module over the ring D includes a basic submodule.*

Proof. Let $y \in P \setminus P^2$. Proposition 6.2 implies that $AP^k = Ay^k$ for every $k \in \mathbb{N}$. Put

$$A_0 = \Omega_{P,1}(A),\ A_k = \Omega_{P,1}(Ay^k),\ k \in \square.$$

Then we can think of A_0 as a vector space over the field D/P. Therefore $A_k = C_{k+1} \oplus A_{k+1}$ for some submodule $C_{k+1} = \oplus_{\lambda \in \Lambda(k+1)} c_\lambda D$. If $\lambda \in \Lambda(k+1)$, then $c_\lambda \in Ay^k$. There exists an element a_λ such that $a_\lambda y^k = c_\lambda$. Put $\Lambda = \cup_{k \in N} \Lambda(k)$, $B = \sum_{\lambda \in \Lambda} a_\lambda D$. Now we can apply the similar arguments to those that we used in the proof of

Theorem 6.3 and prove the equation $B = \oplus_{\lambda \in \Lambda} a_\lambda D$. Since $\oplus_{k \in N} C_k = \Omega_{P,1}(B)$. Lemma 9.2 implies that B is a pure submodule of A.

Finally, let $a + B \in \Omega_{P,1}(A/B)$. Then $ay \in B$. Since B is a pure submodule, $ay = by$ for some element $b \in B$. Put $e = a - b$, then $a + B = e + B$ and $ey = 0$, that is $e \in A_0$. From the equation

$$A_0 = C_1 \oplus \ldots \oplus C_k \oplus A_k$$

It follows that $e = e_1 + e_2$ where $e_1 \in C_1 \oplus \ldots \oplus C_k$, $e_2 \in A_k$. In particular, $e_2 \in Ay^k$. Since $(A/B)y^k = (Ay^k + B)/B$, $a + B = e + B = e_2 + B \in (Ay^k + B)/B = (A/B)y^k$. So $\Omega_{P,1}(A/B) \leq (A/B)y^k$ for each $k \in \mathbf{N}$. Lemma 9.3 shows that A/B is divisible.

9.5. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A and B be P – modules over the ring D, $A = \oplus_{\lambda \in \Lambda} a_\lambda D$, $B = \oplus_{\mu \in M} b_\mu D$. Then A and B are isomorphic if and only if $A/AP^k \cong B/BP^k$ for all $k \in \mathbf{N}$.*

Proof. Put

$$\Lambda_k = \{ \lambda \in \Lambda \mid Ann_D(a_\lambda) = P^k \}, \; M_k = \{ \mu \in M \mid Ann_D(b_\mu) = P^k \},$$
$$A_k = \oplus_{\lambda \in \Lambda k} a_\lambda D, \; B_k = \oplus_{\mu \in Mk} b_\mu D, \; k \in \mathbf{N}.$$

Then $A = \oplus_{k \in \square} A_k$, $B = \oplus_{k \in \square} B_k$. It follows that A and B are isomorphic if and only if $A_k \cong B_k$ for every $k \in \mathbf{N}$. In addition, the submodules A_k and B_k are isomorphic if and only if $|\Lambda_k| = |M_k|$.

Let $k \in \mathbf{N}$, $r_k = min\{ n > k \mid \Lambda_n \neq \varnothing \}$. If $\Lambda_1 = \ldots = \Lambda_k = \varnothing$, then $A/AP^{r}k$ is the direct sum of submodules, every of which is isomorphic with $D/P^{r}k$. In particular, $B/BP^{r}k$ has no a direct summand which is isomorphic to D/P^m for $1 \leq m \leq k$. This means that $M_1 = \ldots = M_k = \varnothing$.

Suppose that $\Lambda_k \neq \varnothing$. If $\Lambda_n = \varnothing$ for each $n > k$, then A/AP^n is a direct sum of submodules which are isomorphic to D/P^j, $1 \leq j \leq k$. From $A/AP^n \cong B/BP^n$ we obtain that $BP^k \leq BP^n$ for any $n > k$. It follows that $BP^k \leq \cap_{n>k} BP^n = < 0 >$. This means that $BP^k = < 0 >$. Therefore $B \cong B/BP^k \cong A/AP^k \cong A$.

Finally assume that there exists the number r_k. Then $A/AP^{r}k \cong C_1 \oplus E$ where $C_1 \cong A_1 \oplus \ldots \oplus A_k$, E is a direct sum of isomorphic copies of $D/P^{r}k$. From $A/AP^{r}k \cong B/BP^{r}k$ we obtain that $|\Lambda_1| = |M_1|, \ldots, |\Lambda_k| = |M_k|$.

9.6. Theorem. *Let D be a Dedekind domain, $P \in Spec(D)$, A a P – module over the ring D. If B, C are the basic submodules of A, then B and C are isomorphic.*

Proof. Since A/B is a divisible module, $(A/B) = (A/B)y^k = (Ay^k + B)/B$ for every $k \in \mathbf{N}$, where $y \in P \setminus P^2$. In particular, $A = Ay^k + B$, therefore

$$A/Ay^k = (Ay^k + B)/Ay^k \cong B/(B \cap Ay^k) = B/By^k$$

for every $k \in \mathbf{N}$. Proposition 6.2 implies that $Ay^k = AP^k$, $By^k = BP^k$, so $A/AP^k \cong B/BP^k$. Similarly, $A/AP^k \cong C/CP^k$. Hence $C/CP^k \cong B/BP^k$ for every $k \in \mathbf{N}$. Lemma 9.5 now implies that $B \cong C$.

We can extend on arbitrary modules the concept of the basis submodule.

Let D be a Dedekind domain, $P \in Spec(D)$, A a D–module. A submodule B is called a P – *basic submodule* if it satisfies the following conditions :

(1) $B = B_1 \oplus B_2$ where B_1 is a direct sum of cyclic P – submodules, B_2 is a projective D – submodule;

(2) $(A/B)P = A/B$;

(3) $B \cap AP^n = BP^n$ for any $n \in \mathbf{N}$.

For construction of P – basis submodules we need the following additional results.

9.7. Lemma. *Let D be a Dedekind domain, L an ideal of D, A a D – module,*

$$<0> = A_0 \leq A_1 \leq \ldots A_\alpha \leq A_{\alpha+1} \leq \ldots A_\gamma = A$$

an ascending series of submodules. If $A_{\alpha+1}/A_\alpha = (A_{\alpha+1}/A_\alpha)L$ for any $\alpha < \gamma$, then $A = AL$.

Proof. We will use induction on α. If $\alpha = 1$, then the assertion is trivial. Let $\alpha > 1$ and assume that we have already proved the equation $(A_\beta)L = A$ for all $\beta < \alpha$. If α is a limit ordinal, then

$$A_\alpha = \bigcup_{\beta<\alpha} A_\beta = \bigcup_{\beta<\alpha} A_\beta L = A_\alpha L.$$

If there exists $\alpha - 1$, then the submodule $B = A_{\alpha-1}$ satisfies the condition $B = BL$. Let $a \in A_\alpha$. From $(A_\alpha/B) = (A_\alpha/B)L = (A_\alpha L + B)/B$ it follows that

$$a = a_1 x_1 + \ldots + a_n x_n + b \text{ where } a_1, \ldots, a_n \in A_\alpha,\ x_1, \ldots, x_n \in L,\ b \in B.$$

Since $B = BL$, $b = b_1 z_1 + \ldots + b_t z_t$ where $b_1, \ldots, b_t \in B, z_1, \ldots, z_t \in L$. Thus

$$a = a_1 x_1 + \ldots + a_n x_n + b_1 z_1 + \ldots + b_t z_t \in A_\alpha L.$$

It follows that $A_\alpha = A_\alpha L$. For $\alpha = \gamma$ we obtain that $A = AL$.

9.8. Lemma *Let D be a Dedekind domain, L an ideal of D, A a D-module, B and C the submodules of A, $B \geq C$. If $C \cap AL = CL$, $(A/C)L \cap (B/C) = (B/C)L$, then $AL \cap B = BL$.*

Proof. Let $b \in B \cap AL$. Then

$$b + C \in (AL + C/C) \cap (B/C) = (A/C)L \cap (B/C) = (B/C)L = (BL + C)/C,$$

that is $b = b_1x_1 + \ldots + b_nx_n + c$ for some $b_1, \ldots, b_n \in B$, $c \in C$, $x_1, \ldots, x_n \in L$. It follows that

$$c = b - b_1x_1 - \ldots - b_n x_n \in AL, \text{ so } c \in AL \cap C = CL.$$

Hence $b \in BL + CL \leq BL$, that is $B \cap AL \leq BL$.

9.9. Corollary. *Let D be a Dedekind domain, L an ideal of D, A a D – module, B a submodule having an ascending series of submodules*

$$< 0 > = B_0 \leq B_1 \leq \ldots B_\alpha \leq B_{\alpha+1} \leq \ldots B_\gamma = B.$$

If $(B_{\alpha+1}/B_\alpha) \cap (A/B_\alpha)L = (B_{\alpha+1}/B_\alpha)L$ for every $\alpha < \gamma$, then $B \cap AL = BL$.

Proof. We will use the induction on α. If $\alpha = 1$, then the assertion is true. Assume that $\alpha > 1$ and we have already proved that $AL \cap B_\beta = B_\beta L$ for every $\beta < \alpha$ If α is a limit ordinal, then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ so

$$AL \cap B_\alpha = AL \cap (\bigcup_{\beta<\alpha} B_\beta) = \bigcup_{\beta<\alpha} (AL \cap B_\beta) = \bigcup_{\beta<\alpha} B_\beta L = B_\alpha L.$$

If there exists $\alpha - 1$, then $B_{\alpha-1} \cap AL = B_{\alpha-1}L$ by the induction hypothesis. Lemma 9.8 implies that $B_\alpha \cap AL = B_\alpha L$. For $\alpha = \gamma$ we obtain the equation $BL = B \cap AL$.

9.10. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A a torsion – free D – module and $r_0(A) = 1$. If B is a finitely generated non – zero submodule of A, then either the P – component of A/B is a cyclic P – module or it is a Prüfer P – module.*

Proof. Let F be the field of fractions of the ring D, $0 \neq b \in B$. If $bx = by$ for some elements $x, y \in D$, then $0 = bx - by = b(x - y)$. Since A is torsion – free, $x - y = 0$, i. e. $x = y$. Therefore if we put $(bx)\phi = x$ $1 = x$, $x \in D$, then ϕ is a mapping. It is easy to see that ϕ is a D – homomorphism and $Im\ \phi = D$. By Proposition 5.8, F is an injective D – module, therefore exists a D – homomorphism Φ: $A \longrightarrow F$ such that $(bx)\Phi = (bx)\phi$ for every $x \in D$. Let $a \in Ker\ \Phi$. Since $r_0(A) = 1$, A/bD is a D – periodic module. In particular, $ax_1 \in bD$ for some $0 \neq x_1 \in D$. In other words, $ax_1 = bx_2$ for some $x_2 \in D$. Now we obtain the following equalities

$$0 = (a\Phi)x_1 = (ax_1)\Phi = (bx_2)\Phi = (b\,x_2)\,\phi = x_2.$$

It follows that $ax_1 = 0$, so $a = 0$ because A is torsion – free. Hence Φ is a monomorphism.

Let R/D be the P – component of F/D, $R_1/D = \Omega_{P,1}(R/D)$. If $y \in P \setminus P^2$, then $R_1y \leq D$. The defined by the rule $x\ \psi = xy$, $x \in R_1$ mapping ψ: $R_1 \longrightarrow D$, is a D –

homomorphism of R_1 in D. Since F does not have zero-divisors, then $Ker\ \psi = < 0 >$, that is $R_1 \cong Im\ \psi = L$. Moreover, L is an ideal of D. By Corollary 1.28, $L/LP \cong D/P$, and thus $R_1/(R_1)P \cong D/P$. It follows that $R_1/D \cong D/P$. Lemma 7.12 implies that R/D is an artinian D – module, and Corollary 7.10 shows that $R/D \cong \mathbf{C}_{p^\infty}$. The factor – module A/bD is isomorphic with some submodule of F/D, thus the P – component of A/bD is isomorphic with a submodule of $\mathbf{C}_{p^\infty}$, i.e. either it is a cyclic submodule or it is isomorphic to module $\mathbf{C}_{p^\infty}$.

9.11. Corollary. *Let D be a Dedekind domain, $P \in Spec(D)$, A a torsion-free D-module, and $r_0(A) = 1$. Then either there is a non-zero finitely generated submodule B such that A/B is periodic and $P \notin Ass_D(A/B)$, or the P-component of A/C is isomorphic to $\mathbf{C}_{p^\infty}$ for every non-zero finitely generated submodule C.*

Proof. If for every finitely generated submodule C the P – component of A/C is isomorphic to $\mathbf{C}_{p^\infty}$, then the statement is proved. Suppose that for some finitely generated submodule U the P – component B/U of A/U is not isomorphic to $\mathbf{C}_{p^\infty}$. By Lemma 9.10, this means that B/U is cyclic. It follows that B is finitely generated. Since B/U is the P – component of A/U, $P \notin Ass_D(A/B)$.

9.12. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A a torsion – free D – module, and $r_0(A) = 1$. Then either $A = AP$ or A includes a finitely generated submodule B such that $(A/B)P = A/B$ and $B \cap AP^n = BP^n$ for any $n \in \mathbf{N}$.*

Proof. Assume firstly, that the P– component of A/C is a Prüfer P - module for every non – zero finitely generated submodule C. Let $0 \neq a \in A$, R/aD be the P – component of A/aD, $0 \neq x \in P$, $a_1 \in R$. Put $a_2 = a_1 x$, $C = a_2 D$. Then $R/C = U_1/C \oplus U_2/C$ where U_2/C is the P – component of R/C and U_1/C is the P' – component of R/C. Clearly, U_1/C is finitely generated and D-periodic. Since $P \notin Ass_D(A/R)$, U_2/C is the P – component of A/C. It follows that U_2/C is a Prüfer P – module, in particular, $(U_2/C)x = U_2/C$ because a Prüfer P – module is divisible by Lemma 5.20. Let $a_1 + C = (u_1 + C) + (u_2 + C)$ where $u_1 \in U_1$, $u_2 \in U_2$. Then $u_2 = u_3 x + c_1$ where $c_1 \in C$. In turn $c_1 = a_2 z_1$ for some $z_1 \in D$, that is $c_1 = (a_1 x) z_1 = (a_1 z_1)x$. Hence

$$u_2 = u_3 x + c_1 = u_3 x + (a_1 z_1)x = (u_3 + a_1 z_1)x = u_4 x$$

and $u_4 \in R$. Since $P \notin Ass_D(U_1/C)$, Lemma 6.7 yields that $U_1/C = (U_1/C)P$. In other words,

$$u_1 + C = (v_1 + C)x_1 + \ldots + (v_m + C)x_m$$

for some elements $v_1, \ldots, v_m \in U_1$, $x_1, \ldots, x_m \in P$. Then

$$u_1 = v_1 x_1 + \ldots + v_m x_m + c_2 \text{ where } c_2 \in C.$$

We have again that $c_2 = a_2 z_2 = (a_1 x) z_2 = a_1 z_2 x$ for some element $z_2 \in D$, so that $u_1 = v_1 x_1 + \ldots + v_m x_m + a_1 z_2 x \in RP$.

Finally, $a_1 = u_1 + u_2 + c_3$ where $c_3 \in C$, thus $c_3 = a_2 z_3 = (a_1 x) z_3 = a_1 z_3 x$ for some element $z_2 \in D$. This means that $a_1 \in RP$, hence $R = RP$. Since A/R is D-periodic and $P \notin Ass_D(A/R)$, Lemma 6.7 implies that $(A/R)P = A/R$. By $R = RP$ and $(A/R)P = A/R$, applying Lemma 9.7 we obtain that $A = AP$.

Suppose now that there is a finitely generated submodule B such that A/B is a D – periodic module with the property $P \notin Ass_D(A/R)$. Put $B_1 = BP^n$. Then B/B_1 is the P – component of A/B_1. Thus $A/B_1 = B/B_1 \oplus E/B_1$ where E/B_1 is the P – component of A/B_1. It follows that $(A/B_1)P^n \leq E/B_1$. On the other hand, by Lemma 6.7 we have that $(E/B_1)P^n = E/B_1$. Consequently,

$$E/B_1 = E/BP^n = (A/BP^n)P^n = (AP^n + BP^n)/BP^n = AP^n/BP^n.$$

It follows that $AP^n \cap B = E \cap B = BP^n$.

9.13. Lemma. *Let D be a Dedekind domain, L a non-zero ideal of D, A a D – module, B a submodule of A. If A/B is D-torsion-free then $B \cap AL = BL$.*

Proof. Let $b \in B \cap AL$. Since $A^* = A/B$ is torsion – free, $B^* = B/BL$ is its D – periodic part. By Proposition 8.9 there exists a submodule C^* such that $A^* = B^* \oplus C^*$. Then $A^*L = B^*L \oplus C^*L = C^*L \leq C^*$. In particular, $A^*L \cap B^* = < 0 >$, so $b + BL = BL$. Thus $B \cap AL = BL$.

9.14. Proposition. *Let D be a Dedekind domain, $P \in Spec(D)$, A a torsion – free D – module. Then A has a P – basic submodule.*

Proof. Proposition 8.17 implies that A has an ascending series of submodules

$$< 0 > = A_0 \leq A_1 \leq \ldots A_\alpha \leq A_{\alpha+1} \leq \ldots A_\gamma = A$$

such that A/A_α is D– torsion–free and $r_0(A_{\alpha+1}/A_\alpha) = 1$ for every $\alpha < \gamma$.

We will apply induction on α. Moreover, we will prove that A_α includes a submodule B_α satisfying the following conditions:

$B_\alpha \cap A_{\alpha-1} = B_{\alpha-1}$ (in particular, $B_{\alpha-1} \leq B_\alpha$);

B_α and $B_\alpha/B_{\alpha-1}$ are projective modules;

$(A_\alpha/B_\alpha)P = A_\alpha/B_\alpha$;

$B_\alpha \cap A_\alpha P^n = B_\alpha P^n$ for any $n \in \mathbf{N}$.

If $\alpha = 1$, then we can employ Lemma 9.12. Suppose $\alpha > 1$ and we have already constructed the submodules B_β for all $\beta < \alpha$.

If α is a limit ordinal, then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. In this case put $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$. It is easy to see that $B_\alpha \cap A_\beta = B_\beta$ for $\beta < \sigma < \alpha$. Therefore $B_\alpha \cap A_\beta = B_\beta$. Lemma 4.17 yields that the submodule B_α is D – projective. Furthermore,

$$(A_\beta + B_\alpha)/B_\alpha \cong A_\beta/(A_\beta \cap B_\alpha) = A_\beta/B_\beta.$$

in particular, $((A_\beta + B_\alpha)/B_\alpha)P = (A_\beta + B_\alpha)/B_\alpha$ for every $\beta < \alpha$. It follows that $(A_\alpha/B_\alpha)P = (A_\alpha/B_\alpha)$ by Lemma 9.7. Finally, Corollary 9.9 implies that $B_\alpha \cap A_\alpha P^n = B_\alpha P^n$.

Now assume that $\alpha - 1$ exists. Then $A_{\alpha-1}$ includes a projective D – submodule $B_{\alpha-1}$ such that

$$(A_{\alpha-1}/B_{\alpha-1})P = A_{\alpha-1}/B_{\alpha-1} \text{ and } B_{\alpha-1}P^n = B_{\alpha-1} \cap A_{\alpha-1}P^n \text{ for each } n \in \mathbf{N}.$$

From Lemma 9.12 it follows that $A_\alpha/A_{\alpha-1}$ includes a finitely generated submodule $C/A_{\alpha-1}$ such that

$$(A_\alpha/C)P = A_\alpha/C,\ (C/A_{\alpha-1}) \cap (A_\alpha/A_{\alpha-1})P^n = (C/A_{\alpha-1})P^n$$

for each $n \in \mathbf{N}$. Hence $C/B_{\alpha-1} = A_{\alpha-1}/B_{\alpha-1} \oplus B_\alpha/B_{\alpha-1}$ and B_α is a projective submodule by Lemma 4.17. By the isomorphism $C/B_\alpha \cong A_{\alpha-1}/B_{\alpha-1}$, we have $(C/B_\alpha)P = C/B_\alpha$. From the equality $(A_\alpha/C)P = A_\alpha/C$ and Lemma 9.7 it follows that $(A_\alpha/B_\alpha)P = (A_\alpha/B_\alpha)$.

Finally, assuming that $b \in B_\alpha \cap A_\alpha P^n$, we have

$$b + A_{\alpha-1} \in (A_\alpha P^n + A_{\alpha-1}/A_{\alpha-1}) \cap (B_\alpha + A_{\alpha-1}/A_{\alpha-1}) = (A_\alpha/A_{\alpha-1})P^n \cap (C/A_{\alpha-1}) = (C/A_{\alpha-1})P^n = (B_\alpha + A_{\alpha-1}/A_{\alpha-1})P^n = (B_\alpha P^n + A_{\alpha-1})/A_{\alpha-1}.$$

In other words, $b = b_1 x_1 + \ldots + b_s x_s + e$ where $b_1, \ldots, b_s \in B_\alpha$, $x_1, \ldots, x_s \in P^n$, $e \in A_{\alpha-1}$. Furthermore, $e = b - b_1 x_1 - \ldots - b_s x_s$ that is, $e \in B_\alpha \cap A_{\alpha-1} = B_{\alpha-1}$. Thus $e \in B_{\alpha-1} \cap A_\alpha P^n$. Since $A_\alpha/A_{\alpha-1}$ is D – torsion – free, Lemma 9.13 yields that $A_{\alpha-1} \cap A_\alpha P^n = A_{\alpha-1}P^n$. So

$$e \in B_{\alpha-1} \cap A_\alpha P^n = B_{\alpha-1} \cap A_{\alpha-1} \cap A_\alpha P^n = B_{\alpha-1} \cap A_{\alpha-1}P^n = B_{\alpha-1}P^n.$$

It follows that $b \in B_\alpha P^n$.

9.15. Theorem. *Let D be a Dedekind domain, $P \in Spec(D)$, A a D – module. Then A has a P – basic submodule.*

Proof. Let T be a D – periodic part of A, R a P – component of A. Lemma 8.5 yields that R is a pure submodule of A. It follows by Theorem 9.4 that R induces the basis submodule B_1. Lemma 8.4 implies that B_1 is a pure submodule of A. By Corollary 3.8 $T = R \oplus Q$ where Q is the P' - component of A. Let $y \in P \setminus P^2$. Proposition 8.2 implies that $RP^n = Ry^n$ and $B_1P^n = B_1y^n$, so we have

$$B_1 \cap RP^n = B_1 \cap Ry^n = B_1y^n = B_1P^n.$$

By Lemma 6.7, $QP = Q$; in particular, $QP^n = Q$. It follows that

$$TP^n = RP^n \oplus QP^n = RP^n \oplus Q.$$

Hence

$$B_1 \cap TP^n = B_1 \cap R \cap TP^n = B_1 \cap (R \cap (RP^n \oplus Q) =$$
$$B_1 \cap (RP^n \oplus (R \cap Q)) = B_1 \cap RP^n = B_1P^n.$$

By Lemma 9.13 $T \cap AP^n = TP^n$, so that

$$B_1 \cap AP^n = (B_1 \cap T) \cap AP^n = B_1 \cap (T \cap AP^n) = B_1 \cap TP^n = B_1P^n.$$

From Proposition 9.14 we obtain that A/T includes the P – basic submodule B_2/T. Since B_2/T is projective, Theorem 4.12 implies that $B_2/B_1 = T/B_1 \oplus B/B_1$ for some submodule B, moreover, $B/B_1 \cong B_2/T$. In particular, B/B_1 is a projective submodule. By Theorem 4.12 $B = B_1 \oplus B_3$ where $B_3 \cong B/B_1$ is a projective module. Thus B_1 is a direct sum of cyclic P – submodules, B_3 is a projective submodule. Let

$$b + B_1 \in (B/B_1) \cap (A/B_1)P^n = (B/B_1) \cap (AP^n + B_1)/B_1.$$

Then $b + B_1 = a_1 x_1 + \ldots + a_s x_s + B_1$ where $a_1, \ldots, a_s \in A$, $x_1, \ldots, x_s \in P^n$. It follows that

$$b + T = a_1 x_1 + \ldots + a_s x_s + T \in (AP^n + T/T) \cap (B_2/T) = (A/T)P^n \cap (B_2/T).$$

By the selection of B_2 we obtain that

$$(A/T)P^n \cap (B_2/T) = (B_2/T)P^n = (B_2P^n + T)/T.$$

Hence $b + T = b_1z_1 + \ldots + b_t z_t + T$ where $b_1, \ldots, b_t \in B_2$, $z_1, \ldots, z_t \in P^n$. Since $B_2/B_1 = (T/B_1) \oplus (B/B_1)$, $b_j + T = c_j + T$ for some element $c_j \in B$, $1 \leq j \leq t$. So we have $b = c_1z_1 + \ldots + c_t z_t + u$ for some element $u \in T$. Then $u = b - c_1z_1 - \ldots - c_t z_t$, so that $u + B_1 \in (B/B_1) \cap (T/B_1) = < 0 >$. It follows that $u \in B_1$ and hence $b + B_1 \in (B/B_1)P^n$. Finally, by the equations $(A/B_1)P^n \cap (B/B_1) = (B/B_1)P^n$, $A \cap B_1P^n = B_1P^n$, applying Lemma 9.8 we obtain that $A \cap BP^n = BP^n$. Since R/B_1 is a divisible module, $(R/B_1)x = R/B_1$ for every $0 \neq x \in P$. It follows that $(R/B_1)P = R/B_1$. Since $T/B_1 = (R/B_1) \oplus (Q + B_1/B_1)$ and $(Q + B_1)/B_1 \cong Q$, $(Q + B_1/B_1)P = Q + B_1/B_1$ by Lemma 6.7. It follows that $T/B_1 = (T/B_1)P$. By the isomorphism $B_2/B \cong T/B$ we obtain that $(B_2/B)P = B_2/B$. Since $(A/B_2)P = A/B_2$, Lemma 9.7 proves that $(A/B)P = A/B$. This means that B is a P – basic submodule of A.

Chapter 10

$\mathfrak{C}_0$ - layer - finite Modules over the Dedekind Domains

This chapter is dedicated to an notable important type of modules that quite frequently appear in various investigations of modules and groups with finiteness conditions.

If G is a group and n is a positive integer, then the subgroup

$$G[n] = < g \in G \mid | g | = n >$$

is called the *nth layer* of the group G. The groups whose layers are finite (*layer – finite groups*) have been introduced by R. Baer [**BR1948**] and S.N. Chernikov [**CS1949**] and have been fully described in [**CS1949, CS1958**].

Dual to the concept of *nth* layer is the concept of *co – layer*. If G is a group and n is a positive integer, then the factor – group G/G^n is called the *nth co – layer* of the group G. The groups G is said to be *co – layer – finite groups*, if the factor – group G/G^n is finite for each $n \in \mathbb{N}$. These groups appear in various group theoretical investigations. L.A. Kurdachenko introduced these groups and began their study in [KL1982]. The class of co – layer – finite groups is quite large. In particular, it includes all layer – finite groups and groups of assorted finite ranks.

In [FdeGK1996] a naturally introduced module analog of co – layer – finite groups has played a key role.

Let R be a ring. An R – module A is called *co – layer – finite*, if for every non – zero ideal L of R the factor – module A/AL is finitely generated as an R – module.

In [FdeGK1996] the co – layer – finite modules have been described in the case when R is a principal domain. In this chapter we consider the structure of co – layer – finite modules over Dedekind domains.

10.1. Lemma. *Let D be a Dedekind domain and let A be a D-module. Then A is co – layer – finite if and only if A/AP^n is finitely generated, for any $P \in Spec(D)$ and $n \in \mathbb{N}$.*

Proof. Suppose that A/AP^n is finitely generated, for every $P \in Spec(D)$ and every $n \in \mathbb{N}$. Given a non – zero ideal L of D, by Theorem 1.14 and Proposition 1.17, we have

$$L = P_1^{n_1} \dots P_k^{n_k} = P_1^{n_1} \cap \dots \cap P_k^{n_k}$$

where the $P_1, \dots P_k$ are different non-zero prime ideals of D. Consider the factor-module $A^* = A/AL$. Applying Corollary 3.8 we obtain

$$A^* = A^*_1 \oplus \dots \oplus A^*_k$$

where A^*_j is the P_j – component of A^*, $1 \leq j \leq k$. Since by Lemma 6.7 $(A^*_j) P_m^{n_m} = A^*_j$ whenever $j \neq m$, $(A^*)P_j^{n_j} = \oplus_{m \neq j} A^*_m$. Since $A/A P_j^{n_j}$ is finitely generated, we have that

$$A^*_j = A^*/(A^*)P_j^{n_j} = (A/AL)/(A/AL)P_j^{n_j} = (A/AL)/((AP_j^{n_j} + AL)/AL) = (A/AL)/((AP_j^{n_j})/AL) \cong A/AP_j^{n_j}$$

is finitely generated giving that A/AL is finitely generated too. The converse statement is obvious.

10.2. Lemma. *Let D be a Dedekind domain and let A be a D – module. Then A is co – layer – finite if and only if A/Ax is finitely generated, for every $0 \neq x \in D$.*

Proof. The direct assertion is a trivial application of Lemma 10.1. For the converse, it suffices to recall that by Corollary 1.26 every ideal of a Dedekind ring can be generated by at most two elements.

10.3. Lemma. *Let D be a Dedekind domain and let A be a D – module and B a D -- submodule of A.*

(1) If A is co – layer – finite, then so is A/B.

(2) If B and A/B are co – layer – finite, then A is also co – layer – finite.

Proof. (1) is obvious.

(2) Let L be a non - zero ideal of D. Then $(A/B)/((AL + B)/B)$ is finitely generated. Since

$$(AL + B)/AL \cong B/(B \cap AL) \text{ and } BL \leq (B \cap AL),$$

$(AL + B)/AL$ is finitely generated too. The isomorphism

$$(A/(AL))/((AL + B)/AL) \cong A/(AL + B)$$

implies that A/AL is finitely generated.

10.4. Lemma. *Let D be a Dedekind domain and let A be a D – module and B a pure D – submodule of A. If A is co – layer – finite, then so is B.*

Proof. Let $0 \neq x \in D$, then $B \cap Ax = Bx$, so that

$$B/Bx = B/(B \cap Ax) \cong (Ax + B)/Ax \leq A/Ax.$$

Since D is a noetherian ring and A/Ax is finitely generated, then B/Bx is finitely generated too. It suffices to apply Lemma 10.2.

10.5. Proposition. *Let D be a Dedekind domain, P a maximal ideal of D, A a P – module over the ring D. If $\dim_{D/P} A/AP$ is finite, then $A = B \oplus C$ where B is a finitely generated submodule, C is a divisible submodule.*

Proof. Let

$$A/AP = \bigoplus_{1 \leq j \leq n} (a_j D + AP),\ A_0 = \sum_{1 \leq j \leq n} a_j D.$$

Then $A = A_0 + AP$. Since A_0 is a finitely generated P -- module, there is a number s_1 such that $P^{s_1} \leq Ann_D(A_0)$. Therefore there exists a number s_2 such that $A_0 y^{s_2} = <0>$ but $A_0 y^{s_2 - 1} \neq <0>$. Put $C = Ay^{s_2}$, then

$$C = A y^{s_2} = (A_0 + AP)y^{s_2} = (A_0 + Ay)y^{s_2} =$$
$$A_0 y^{s_2} + A y^{s_2 + 1} = A y^{s_2 + 1} = (Ay^{s_2})y = Cy.$$

Proposition 6.2 implies that C is a divisible submodule, therefore Theorem 5.13 gives a direct decomposition $A = C \oplus B$ for some submodule B. Furthermore

$$A = A_0 + AP = A_0 + (A_0 + AP)P = A_0 + AP^2 = \ldots = A_0 + AP^{s(2)} = A_0 + C.$$

Now $B \cong A/C = (A_0 + C)/C \cong A_0/(A_0 \cap C)$ is a finitely generated submodule.

10.6. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A a co – layer – finite D – module. If A is a P-module, then there exists a direct decomposition $A = B \oplus C$, where B is finitely generated and C is D – divisible.*

Indeed, since A/AP is finitely generated, $dim_{D/P}(A/AP)$ is finite. Now we may apply Proposition 10.5.

10.7. Corollary. *Let D be a Dedekind domain and let A be a periodic D – module. If A is a co – layer – finite, then*

$$A = \bigoplus_{P \in \pi} A_P$$

where $\pi = \mathrm{Ass}_D(A)$, $A_P = B_P \oplus C_P$ is the P-component of A, B_P is finitely generated and C_P is divisible.

10.8. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A a D – module. Suppose that A is D – torsion – free and $r_0(A)$ is finite. Let B be a finitely generated submodule of A such that A/B is D – periodic. If Y/B is the P – component of A/B, then*

$$Y/B = C_1 \oplus \ldots \oplus C_k \oplus E_1 \oplus \ldots \oplus E_n$$

where C_i is a Prüfer P – module, $1 \le i \le k$, E_j is a cyclic submodule, $1 \le j \le n$.

Proof. Let $Y_1/B = \Omega_{P,1}(Y/B)$. Then $Y_1P \le B$. Let $z \in P \setminus P^2$. Consider the mapping

$$\phi_z: a \longrightarrow az, \; a \in Y_1.$$

Clearly $Im\ \phi_z \le B$. Since A is D – torsion – free, $Ker\ \phi_z = < 0 >$, in particular, $Y_1 \cong Im\ \phi_z$. Hence $Y_1/B \cong (Im\ \phi_z)/(B\phi_z)$. It follows from the inclusion $Im\ \phi_z \le B$ that $\mathrm{Im}\ \phi_z$ is finitely generated, thus Y_1/B is finitely generated too. This means that $dim_{D/P}(Y_1/B)$ is finite. It follows that Y/B is artinian D – module by Corollary 7.12. Theorem 7.13 proves that

$$Y/B = C_1 \oplus \ldots \oplus C_k \oplus E_1 \oplus \ldots \oplus E_n$$

where C_i is a Prüfer P – module, $1 \le i \le k$, E_j is a cyclic submodule, $1 \le j \le n$.

10.9. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A a D – module, B a submodule of A such that A/B is D – torsion – free, and $(A/B)P = A/B$. Then $A/AP^n \cong B/BP^n$ for every $n \in \mathbb{N}$.*

Proof. Since A/B is torsion-free then $B^\sharp = B/BP^n$ is the D – periodic part of $A^\sharp = A/BP^n$. By Proposition 8.9, there is a D – submodule $C^\sharp$ such that $A^\sharp = B^\sharp \oplus C^\sharp$. Moreover,

$$C^\sharp \cong A^\sharp/B^\sharp = (A/BP^n)/(B/BP^n) \cong A/B.$$

In particular, $(C^\sharp)P^n = C^\sharp$. It follows that

$$(A^\sharp)P^n = (B^\sharp)P^n \oplus (C^\sharp) = C^\sharp.$$

Hence

$$A/AP^n \cong (A/BP^n)/((AP^n)/(BP^n)) = A^\sharp/C^\sharp = (B^\sharp \oplus C^\sharp)/C^\sharp \cong B^\sharp = B/BP^n.$$

10.10. Lemma. *Let D be a Dedekind domain, $P \in Spec(D)$, A a torsion – free D – module of finite D – rank, B a finitely generated submodule of A such that A/B is D – periodic. If B is a P – basic submodule of A then $A/AP^n \cong B/BP^n$ for every $n \in \mathbb{N}$.*

Proof. Let Y/B be the P – component of A/B. Corollary 3.8 yields that $A/B = Y/B \oplus Q/B$, where Q/B is the P'-component of A/B. In particular, $(Q/B)P = Q/B$ by Lemma 6.7. Thus $(A/B)P = (Y/B)P \oplus (Q/B)$. Since $(A/B)P = A/B$, $(Y/B)P = Y/B$. Lemma 10.8 yields that Y/B is a direct sum of finitely many Prüfer P-submodules. Lemma 5.20 implies that Y/B is divisible.

Consider the factor – module A/BP^n. Clearly, Y/BP^n is the P – component of A/BP^n. Lemma 10.8 proves that $Y/BP^n = E/BP^n \oplus C/BP^n$ where E/BP^n is the D – divisible part of Y/BP^n, C/BP^n is a finitely generated submodule. Since B/BP^n is finitely generated,

$$(E/BP^n + B/BP^n)/(B/BP^n) = ((E + B)/BP^n)/(B/BP^n) \cong (E + B)/B$$

is the D-divisible part of Y/B. But Y/B is D-divisible, that is, $(E + B)/B = Y/B$.

Put now $A^\blacktriangle = A/BP^n$, $B^\blacktriangle = B/BP^n$, $E^\blacktriangle = E/BP^n$, $Y^\blacktriangle = Y/BP^n$. Hence $A^\blacktriangle = Y^\blacktriangle \oplus Q^\blacktriangle{}_1$ where $Q^\blacktriangle{}_1$ is the P'- component of $A^\blacktriangle$. We have now

$$A^\blacktriangle/A^\blacktriangle P^n = (A/BP^n)/(A/BP^n)P^n = (A/BP^n)/((A/BP^n + BP^n/BP^n) = (A/BP^n)/(AP^n/BP^n) \cong A/AP^n$$

Lemma 6.7 implies that $Q^\blacktriangle{}_1 P^n = Q^\blacktriangle{}_1$, so that $A^\blacktriangle P^n = Y^\blacktriangle P^n \oplus Q^\blacktriangle{}_1$, that is,

$$A^\blacktriangle/A^\blacktriangle P^n = (Y^\blacktriangle \oplus Q^\blacktriangle{}_1)/(Y^\blacktriangle P^n \oplus Q^\blacktriangle{}_1) \cong Y^\blacktriangle/Y^\blacktriangle P^n$$

Finally, $Y^\blacktriangle = B^\blacktriangle + E^\blacktriangle$, therefore $Y^\blacktriangle P^n = E^\blacktriangle$, thus

$$Y^{\wedge}/Y^{\wedge}P^n = (B^{\wedge} + E^{\wedge})/E^{\wedge} \cong B^{\wedge}/(B^{\wedge} \cap E^{\wedge}) = (B/BP^n)/(B/BP^n \cap E/BP^n) = (B/BP^n)/((B \cap E)/BP^n).$$

Since B is a finitely generated torsion – free module, Corollary 7.6 yields

$$B \cong D \oplus \ldots \oplus D \oplus L$$
$$\text{(} r-1 \text{)}$$

where L is some ideal of D. Since $L/LP^n \cong D/P^n$, it follows that there is a sum of r summands

$$B/BP^n \cong D/P^n \oplus \ldots \oplus D/P^n$$
$$\text{(} r-1 \text{)}$$

Hence if we assume that $B \cap E \neq BP^n$, then $B/(B \cap E)$ has no submodules which is isomorphic with B/BP^n. On the other hand, since B is a P – basic submodule, $B \cap AP^n = BP^n$, so that

$$B/BP^n = B/(B \cap AP^n) \cong (B + AP^n)/AP^n.$$

In particular, A/AP^n (and hence $Y^{\wedge}/Y^{\wedge}P^n$) includes a submodule, which is isomorphic to B/BP^n. This means that $B \cap E = BP^n$ so that $Y^{\wedge}/Y^{\wedge}P^n \cong B/BP^n$. Thus

$$A/AP^n \cong A^{\wedge}/A^{\wedge}P^n \cong Y^{\wedge}/Y^{\wedge}P^n \cong B/BP^n.$$

10.11. Proposition. *Let D be a Dedekind domain, $P \in Spec(D)$, A a torsion – free D – module, B a P – basic submodule of A. If B has finite 0 – rank, then $A/AP^n \cong B/BP^n$ for every $n \in \mathbf{N}$.*

Proof. Let T/B be the D – periodic part of A/B. Since B is P – basic submodule, $(A/B)P = A/B$. Therefore $(A/T)P = A/T$. Lemma 10.9 yields that $A/AP^n \cong T/TP^n$ for any $n \in \mathbf{N}$. Since B is a projective submodule of finite 0 – rank, B is finitely generated. This means that $r_0(T)$ is finite.

Since A/T is torsion – free, Lemma 9.13 implies that $TP^n = T \cap AP^n$. Thus we have

$$B \cap TP^n = B \cap T \cap AP^n = B \cap AP^n = BP^n.$$

Suppose that $(T/B)P \neq T/B$. Let $A^{\wedge} = (A/B)/(T/B)P$, $T^{\wedge} = (T/B)/(T/B)P$. Since $T^{\wedge}$ is the D – periodic part of $A^{\wedge}$ and $P^n \leq Ann_D(T^{\wedge})$, by Proposition 8.9, $A^{\wedge} = T^{\wedge} \oplus C^{\wedge}$ for some D – submodule $C^{\wedge}$. From the selection of B it follows that $A^{\wedge}P = A^{\wedge}$. However,

$$A^{\wedge}P = T^{\wedge}P \oplus C^{\wedge}P = C^{\wedge}P \neq A^{\wedge}.$$

This contradiction shows that $(T/B)P = T/B$. In other words, B is a P-basic submodule of T. Lemma 10.10 implies that in this case, $T/TP^n \cong B/BP^n$, and therefore $A/AP^n \cong T/TP^n \cong B/BP^n$.

10.12. Corollary. *Let D be a Dedekind domain, $P \in Spec(D)$, A a torsion – free D – module. If $\dim_{D/P}(A/AP) = m$ is finite, then*

$$A/AP^n \cong \underbrace{D/P^n \oplus \ldots \oplus D/P^n}_{m}$$

for any $n \in \mathbb{N}$.

Proof. Let B be a P – basic submodule of A. Its existence follows from Theorem 9.15. Since

$$B \cap AP = BP,\ B/BP = B/(AP \cap B) \cong (B + AP)/AP,$$

so that $\dim_{D/P}(B/BP)$ is finite. Assume that B is not finitely generated. Since B is a projective module, Theorem 4.25 shows that B is a free D – module. As a free module of infinite D – rank, the factor – module B/BP is infinite dimensional. Thus B is finitely generated, and we can apply Proposition 10.11.

10.13. Lemma. *Let D be a Dedekind domain and let A be a torsion – free D – module having an ascending chain of D – pure submodules*

$$< 0 > = A_0 \leq A_1 \leq \ldots \leq A_n \leq A_{n+1} \leq \ldots \bigcup_{n \in N} A_n = A$$

such that $r_0(A_{n+1}/A_n) = 1$, for every $n \in \mathbb{N}$. Then A is co – layer – finite if and only if for every $P \in Spec(D)$ there exists a number $h(P) \in \mathbb{N}$ such that $(A_j - 1/A_j)P = (A_{j+1}/A_j)$ for each $j \geq h(P)$.

Proof. Suppose that A is co – layer – finite and assume that there exists $P \in Spec(D)$ such that $(A_{n+1}/A_n)P \neq A_{n+1}/A_n$, for infinitely many numbers n. Then A has an ascending chain of D – pure submodules

$$< 0 > = B_0 \leq B_1 \leq \ldots \leq B_n \leq B_{n+1} \leq \ldots \bigcup_{n \in N} B_n = A$$

such that $r_0(B_{n+1}/B_n)$ is finite and $(B_{n+1}/B_n)P \neq B_{n+1}/B_n$ for every $n \in \mathbb{N}$. In particular, $B_1P \neq B_1$. Consider the factor module B_2/B_1P. Clearly B_1/B_1P is the D – periodic part of B_2/B_1P and, since $Ann_D(B_1/B_1P) = P$, an application of Theorem 8.9 realizes the following direct decomposition

$$B_2/B_1P = B_1/B_1P \oplus C/B_1P$$

for some D – submodule C. Then $C/B_1P \cong B_2/B_1$ and so $(C/B_1P)P \neq C/B_1P$. From the equality

$$(B_2/B_1P)P = (C/B_1P)P = CP/B_1P$$

it follows that $\dim_{D/P}(B_2/B_2P) \geq 2$ and, by induction, it is very easy to establish that $\dim_{D/P}(B_n/B_nP) \geq n$ for every $n \in \mathbf{N}$. Moreover, the equation $A = \bigcup_{n \in N} B_n$ gives that $AP = \bigcup_{n \in N} B_nP$ and so $A/AP = \bigcup_{n \in N}(B_n + AP/AP)$.

Since A/B_n is torsion – free, B_n/B_nP is the periodic part of A/B_nP, and we obtain the decomposition

$$A/B_nP = B_n/B_nP \oplus E/B_nP$$

for certain D – submodule E. It follows that

$$(A/B_nP)P = (E/B_nP)P = EP/B_nP \leq E/B_nP.$$

In particular, $AP \cap B_n = B_nP$ and hence

$$(B_n + AP)/AP \cong B_nP/(B_n \cap AP) = B_n/B_nP,$$

so that $\dim_{D/P}(B_n + AP/AP) \geq n$ for each $n \in \mathbf{N}$. This means that $\dim_{D/P}(A/AP)$ is infinite. Contradiction.

Conversely, suppose that the D – module A satisfies the conditions of the statement. For any $P \in Spec(D)$ we put $h = h(P)$. It follows that $(A_{n+h}/A_h)P = (A_{n+h}/A_h)$ for every $n \in \mathbf{N}$. Consider the factor – module A_{n+h}/A_hP; obviously, A_h/A_hP is its periodic part. Since $Ann_D(A_h/A_hP) = P$, Theorem 8.9 implies the following direct decomposition

$$A_{n+h}/A_hP = A_h/A_hP \oplus C_{n+h}/A_hP$$

for some D – submodule C_{n+h}. Since $C_{n+h}/A_hP \cong A_{n+h}/A_h$, $(C_{n+h}/A_hP)P = C_{n+h}/A_hP$. Thus

$$(A_{n+h}/A_hP)P = A_{n+h}P/A_hP = C_{n+h}/A_hP.$$

This allows us to check that $A_{n+h}/A_{n+h}P \cong A_h/A_hP$. Hence

$$\dim_{D/P}(A_{n+h}/A_{n+h}P) = \dim_{D/P}(A_h/A_hP) = m.$$

Again $A = \bigcup_{n \in \mathbf{N}} A_{n+h}$ gives that $A/AP = \bigcup_{n \in \mathbf{N}}(A_{n+h} + AP)/AP$. Since

$$(A_{n+h} + AP)/AP \cong A_{n+h}/(A_{n+h} \cap AP) = A_{n+h}/A_{n+h}P,$$

$\dim_{D/P}((A_{n+h} + AP)/AP) = m$ for every $n \in \mathbf{N}$. Since

$$A_{n+h} \leq A_{n+h+1},\ (A_{n+h} + AP)/AP = (A_{n+h+1} + AP)/AP,$$

so that $A/AP = (A_h + AP)/AP$, and hence $dim_{D/P}(A/AP) = m$ is finite. By Corollary 10.12, the module A/AP^t is finitely generated, for any $t \in \mathbf{N}$. Thus, Lemma 10.1 implies that A is co – layer – finite, as required.

10.14. Lemma. *Let D be a Dedekind domain such that Spec(D) is countable and let A be a co – layer – finite torsion-free D-module. Then A includes a D – pure submodule B such that $r_0(B)$ is countable and A/B is divisible.*

Proof. Put $Spec(D) = \{ P_n \mid n \in \mathbf{N} \}$. By Theorem 9.15, A includes a P_n – basic submodule B_n for every $n \in \mathbf{N}$. Since B_I is a projective D – module, Theorem 4.25 justifies the equation $B_I = \bigoplus_{\lambda \in A(I)} C_\lambda$ where the C_λ are isomorphic to certain ideals L_λ of D. Then

$$B_I P_I = \bigoplus_{\lambda \in A(I)} C_\lambda P_I \text{ and } B_I/B_I P_I \cong \bigoplus_{\lambda \in A(I)} C_\lambda / C_\lambda P_I.$$

Furthermore, using Corollary 1.28 we obtain $C_\lambda / C_\lambda P_I \cong L_\lambda / L_\lambda P_I \cong D/P_I$. From $B_I \cap A P_I = B_I P_I$ we obtain

$$B_I/B_I P_I \cong B_I/(B_I \cap A P_I) \cong (B_I + AP_I)/AP_I.$$

It follows that $B_I/B_I P_I$ is finitely generated. In other words, $B_I/B_I P_I$ has finite dimension as a vector space over D/P_I. Since this dimension coincides with $|A(I)|$, we conclude that $A(I)$ is a finite set and hence B_I is finitely generated too.

Let A_I/B_I be the periodic part of A/B_I; in particular, $r_0(A_I) = r_0(B_I)$, so that A_I has finite 0 – rank. Moreover A/A_I is a torsion – free D – module and, since $(A/B_I)P_I = A/B_I$, we have that $(A/A_I)P_I = A/A_I$. Repeating these arguments, we construct an ascending chain of D – pure submodules

$$A_I \leq \ldots \leq A_n \leq A_{n+1} \leq \ldots$$

such that $r_0(A_n)$ is finite and $(A/A_n)P_j = A/A_n$, for every j, $1 \leq j \leq n$, $n \in \mathbf{N}$. Applying Lemma 8.1, we deduce that $B = \bigcup_{n \in \mathbf{N}} A_n$ is a D – pure submodule of A, $r_0(B)$ is countable and $(A/B)P_n = A/A_n$, for every $n \in \mathbf{N}$.

Let $0 \neq x \in D$. Since D is a Dedekind domain, Theorem 1.14 implies that

$$xD = \Pi_{P \in \eta} P^{a(P)}$$

where η is a finite subset of $Spec(D)$. $= \{ P_{i1}^{k_1} \ldots P_{ir}^{k_r} \}$. Then

$$(A/B)x = (A/B)(xD) = (A/B) \Pi_{P \in \eta} P^{a(P)},$$

so that A/B is divisible.

It is worth mentioning, that among the Dedekind domains satisfying the above conditions are the ring $\mathbb{Z}$ of all integers and any group – ring of the form $F< x >$ where a field F is countable. It is well known, these rings have many important applications.

10.15. Lemma. *Let D be a Dedekind domain with countable set Spec(D) and let A be a torsion – free D – module. Then A is co – layer – finite if and only if A has an ascending chain of D - pure submodules*

$$< 0 > = A_0 \leq A_1 \leq \ldots \leq A_n \leq A_{n+1} \leq \ldots \bigcup_{n \in \omega} A_n = A_\omega \leq A_{\omega+1} = A$$

such that

(i) A/A_ω is divisibl;

(ii) $r_0 (A_{n+1}/A_n) = 1$, for every $n \in \omega$;

(iii) for every $P \in Spec(D)$ there exists a number $h(P) \in \mathbb{N}$ such that $(A_{n+1}/A_n)P = (A_{n+1}/A_n)$ for each $n \geq h(P)$.

Proof. Suppose first that A is co – layer – finite. By Lemma 10.14 there exists a D – pure submodule of countable 0 – rank such that A/B is divisible. By Proposition 8.17, B has an ascending chain of D – pure submodules

$$< 0 > = A_0 \leq A_1 \leq \ldots \leq A_n \leq A_{n+1} \leq \ldots \bigcup_{n \in N} A_n = B$$

such that $r_0 (A_{n+1}/A_n) = 1$ for every $n \in \mathbb{N}$. Moreover, by Lemma 10.4, B is co – layer – finite. Then Lemma 10.13 shows that for every $P \in Spec(D)$ there exists a number $h(P) \in \mathbb{N}$ such that $(A_{n+1}/A_n)P = (A_{n+1}/A_n)$ for each $n \geq h(P)$.

Conversely, if A satisfies the above conditions, then by Lemma 10.13, the D – submodule A_ω is co – layer – finite. Obviously, A/A_ω is co – layer – finite, since it is divisible. Then A is also co – layer – finite by Lemma 10.3.

10.16. Theorem *Let D be a Dedekind domain with countable set Spec(D) and let A be a D-module. Then A is co – layer – finite if and only if A decomposes as $A = B \oplus E$, and*

(i) D – submodule E is divisible and periodic;

(ii) the submodule B has an ascending chain of D – pure submodules

$$< 0 > = B_0 \leq B_1 \leq \ldots \leq B_n \leq B_{n+1} \leq \ldots B_\omega \leq B_{\omega+1} = B$$

such that B_1 is the D – periodic part of B, B/B_ω is divisible and $r_0 (B_{n+1}/B_n) = 1$ for every $n \in \mathbb{N}$.

(iii) for every $P \in Spec(D)$ there exists a number $h(P) \in \mathbb{N}$ such that $(B_{n+1}/B_n)P = (B_{n+1}/B_n)$ for each $n \geq h(P)$;

(iv) $B_1 = \bigoplus_{p \in \pi} C_p$, where C_p is the P – component of B_1, $\pi = Ass_D(A)$ being each C_p finitely generated.

Proof. Suppose first that A is co – layer – finite. If T is the D – periodic part of A, then by Corollary 10.6 and Lemma 10.4, we may express $T = \oplus_{P \in \pi} T_P$, where $\pi = Ass_D(A)$, $T_P = C_P \oplus E_P$ is the P – component of A, C_P is finitely generated and E_P is divisible. Define $E = \oplus_{P \in \pi} E_P$. So that E is a divisible direct summand of A by Theorem 5.13: $A = E \oplus B$ for some D – submodule B. Consider now $B_l = \oplus_{P \in \pi} C_P$, clearly B_l is the D – periodic part of B. Thus B/B_l is torsion – free and it suffices to apply Lemma 10.15.

Conversely, by Lemma 10.15 B/B_l is co – layer – finite. Clearly, so are B_l and E. Then it suffices to apply Lemma 10.3.

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