Algebra and Discrete Mathematics
Number 2. **(2009).** pp. 70 – 78
© Journal "Algebra and Discrete Mathematics"

RESEARCH ARTICLE

On some generalization of metahamiltonian groups

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Communicated by L. A. Kurdachenko

Abstract. Locally step groups at which all subgroups are or normal, or have Chernikov's derived subgroup are studied.

Let G be a group and $\mathcal{L}_{\rm norm}(G)$ be a family of all normal subgroups of G. Study of the influence of the family $\mathcal{L}_{\rm norm}(G)$ on the structure of the group G was begun a long time ago. R. Dedekind in his classical paper [1] described the finite groups, which subgroups are normal (i.e. the family $\mathcal{L}_{\rm norm}(G)$ consists of all subgroups). Later, R. Baer in a paper [2] extended the description of R. Dedekind on arbitrary groups. Let's note that the groups, any subgroup of which is normal, were later called Dedekind groups. O.Y. Schmidt, in his works [3] and [4], began the study of the (finite) groups, in which the family $\mathcal{L}_{\rm non-norm}(G)$ of all non-normal subgroups satisfies certain strong enough restriction. So in the papers [3] and [4] he studied the structure of finite groups, in which either $\mathcal{L}_{\rm non-norm}(G)$ is a conjugacy class, or is the union of two conjugacy classes. S.N. Chernikov in a paper [5] started the study of infinite groups, in which the family $\mathcal{L}_{\rm non-norm}(G)$ satisfies certain natural finiteness condition. This work caused a large cycle of works devoted to the study of infinite groups, which family $\mathcal{L}_{\rm non-norm}(G)$ satisfies certain natural restriction. Here we won't write in detail about these researches, they are examined in detail in the book of S.N. Chernikov, [6], and the review [7]. Here we will touch upon the works, directly related to this work. G.M. Romalis and N.F. Sesekin in a paper [8-10] started to study the groups, in which the family $\mathcal{L}_{\rm non-norm}(G)$ consists of abelian subgroups. They

2000 Mathematics Subject Classification: *20F22; 20H20.*

Key words and phrases: *group, subgroup, normal subgroup, Chernikov group, locally graded group, locally finite group.*

called such groups metahamiltonian groups. The finite metahamiltonian groups have also been studied in papers [11, 12]. The full description of metahamiltonian groups have been obtained in a series of works of N.F. Kuzennyj and N.N. Semko [13-19]. The natural continuation of such researches is a consideration of a situation, when the subgroups of the family $\mathcal{L}_{\mathrm{non-norm}}(G)$ belong to the class of groups, which are a natural extension of the class of abelian groups. Thus in papers [20, 21] there have been considered the groups, in which the subgroups of a family $\mathcal{L}_{\mathrm{non-norm}}(G)$ have finite derived subgroups or are FC-groups. In this work, the researches in this area continue. Since the Chernikov groups are a natural extension of the finite groups, then the groups with Chernikov derived subgroups are a natural extension of the groups with finite derived subgroups. In this work, we start the study of groups, in which every subgroup is normal or has a Chernikov derived subgroup.

Lemma 1. *Let G be a group whose subgroups either are normal or have Chernikov derived subgroups. Then the following assertions hold:*

1. *If H is a subgroup of G then every subgroup of H either is normal or has Chernikov derived subgroup.*

2. *If L is a normal subgroup of G then every subgroup of G/L either is normal or has Chernikov derived subgroup.*

3. *If S is a subgroup of G such that $[S, S]$ is not Chernikov subgroup, then a factor-group G/S is Dedekind.*

All these assertion are almost obvious.

Lemma 2. *Let G be a group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that A is an abelian torsion-free subgroup of G. If the normalizer $N_G(A)$ contains an element g of finite order, then $g \in C_G(A)$.*

Proof. Suppose the contrary, let $g \notin C_G(A)$. Let m be an arbitrary positive integer and put $H_m = \langle A^m, g\rangle$. Since A^m is a characteristic subgroup of A, A^m is $\langle g\rangle$-invariant, so that $\langle A^m, g\rangle = A^m \leftthreetimes \langle g\rangle$. It implies the inclusion $[H_m, H_m] \leqslant A^m$. Assume that a subgroup $\langle g, A^m\rangle$ is abelian. Then we have $(a^g)^m = (a^m)^g = a^m$ for each element $a \in A$. It follows that $(a^g a^{-1})^m = 1$. Since A is torsion-free, we obtain that $a^g a^{-1} = 1$ or $a^g = a$. This shows that $g \in C_G(A)$, and we obtain a contradiction with our assumption. This contradiction implies the subgroups $\langle g, A^m\rangle$ are non-abelian for all $m \in N$. In turn, it follows that $[H_m, H_m]$ are non-identity for every $m \in N$. An inclusion $[H_m, H_m] \leqslant A^m$ shows

that $[H_m, H_m]$ is torsion-free. In particular, it is not Chernikov. In this case a subgroup $\langle g, A^m\rangle$ is normal for each $m \in N$. Therefore $\langle g\rangle = \cap_{m\in N}\langle g, A^m\rangle$ is likewise normal. An equation $\langle g\rangle \cap A = \langle 1\rangle$ implies that $g \in C_G(A)$, what proves a result. □

Lemma 3. *Let G be a finitely generated group. If G includes a subgroup L of finite index such that $[L, L]$ is a Chernikov subgroup, then G is abelian-by-finite.*

Proof. Put $K = [L, L]$. The finiteness of $|G : L|$ implies that L is finitely generated (see, for example, [22, Theorem 1.41]). Let D be a maximal divisible subgroup (divisible part) of a Chernikov subgroup K. Then L/D includes a normal finite subgroup K/D such that L/K is abelian and finitely generated. In particular, L/D is polycyclic-by-finite. Then L/D is residually finite (see, for example, [23, Chapter 1, Theorem 1]). It follows that L/D includes a normal subgroup F/D of finite index such that $(F/D) \cap (K/D) = \langle 1\rangle$. This shows that F/D is an abelian subgroup of finite index. The finiteness of $|G : F|$ implies that F is finitely generated. Being metabelian and finitely generated, F satisfies a maximal condition on normal subgroups [24, Theorem 3]. On the other hand, if D is non-identity, it has an ascending series of G-invariant subgroups. This contradiction shows that $D = \langle 1\rangle$, i.e. $L \cong L/D$ is abelian-by-finite. The fact that $|G : L|$ is finite proves that G is abelian-by-finite. □

Lemma 4. *Let G be a group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that every subgroup of finite index has non-Chernikov derived subgroup. Let R be an intersection of all subgroup of finite index. Then either G/R is abelian, or G has a normal subgroup $V \geqslant R$ such that G/V is a quaternion group and V/R is abelian.*

Proof. Since every subgroup of finite index of G has non-Chernikov derived subgroup, then every subgroup H of finite index is normal in G. By Lemma 1 G/H is Dedekind group. Let

$$\mathfrak{M} = \{X | X \text{ is a subgroup of finite index in } G\}, \quad R = \cap\mathfrak{M}.$$

By a classical Remak theorem G/R is isomorphic to some subgroup of $\prod_{H\in\mathfrak{M}} G/H$. Suppose that G/R is non-abelian. Then G has a normal subgroup U of finite index such that G/U is a non-abelian Dedekind group. In this case $G/U = Q/U \times V/U$ where Q/U is a quaternion group, V/U is a direct product of an elementary abelian 2-subgroup and periodic abelian subgroup without the elements of order 2 (see, for example, [25, 5.3.7]). Let W be a subgroup of V having finite index. Then W has finite index in G and hence is normal in G. A factor-group G/W is non-abelian

and Dedekind. If we suppose that V/W is non-abelian, then V includes a D-invariant subgroup Y such that V/Y is a quaternion group. Then G/Y is quaternion-by-quaternion. On the other hand, G/Y is Dedekind. This contradiction shows that V/W is abelian. Let

$$\mathfrak{S} = \{X | X \text{ is a subgroup of finite index in } V\}, \quad T = \cap\mathfrak{S}.$$

Clearly $\mathfrak{S}$ is a subfamily of $\mathfrak{M}$, so that $T \geqslant R$. Using again Remak's, we obtain that V/T is isomorphic to a subgroup of $\prod_{W\in\mathfrak{S}} V/W$. Since the last group is abelian, V/T is also abelian. Let $X \in \mathfrak{M}$, then $X \cap V$ has finite index in V, that is $X \cap V \in \mathfrak{S}$. It follows that $\mathfrak{S} = \{X \cap V | X \in \mathfrak{M}\}$. Then

$$R = R \cap V = (\cap_{X\in\mathfrak{M}} X) \cap V = \cap_{X\in\mathfrak{M}}(X \cap V) = T.$$

Hence V/R is abelian. □

Corollary. *Let G be a finitely generated group whose subgroups either are normal or have Chernikov derived subgroups. If G is soluble-by-finite, then G is abelian-by-finite.*

Proof. If G has a subgroup L of finite index whose derived subgroup is Chernikov, then Lemma 3 shows that G is abelian-by-finite. Therefore we can suppose that every subgroup of finite index of G has non-Chernikov derived subgroup. Denote by R the intersection of all subgroup having in G finite index. By Lemma 4 either G/R is abelian, or G includes a normal subgroup V such that $V \geqslant R$, G/V is a quaternion group and V/R is abelian. The finiteness of G/V implies that V is finitely generated. Being finitely generated and abelian-by-finite, V has the proper subgroups of finite index. It follows that $R \neq V$. Since V/R is abelian, $[V, V] = K \neq V$. Let S be a soluble radical of K. Clearly S is G-invariant. Suppose that KS. Then K/S is finite and non-soluble. A factor-group V/S is finite-by-abelian. Being finitely generated, V/S is polycyclic-by-finite. Then V/S is residually finite (see, for example, [23, Chapter 1, Theorem 1]). It follows that V/S has a normal subgroup E/S of finite index such that $(E/S) \cap (K/S) = \langle 1\rangle$. In particular, E has finite index in G, therefore it is normal in G. The relations

$$X/S \cong (X/S)/((E/S) \cap (X/S)) \cong (X/S)(E/S)/(E/S) \leqslant (G/S)/(E/S)$$

and $G/E \cong (G/S)/(E/S)$ show that G/E includes a finite non-soluble subgroup. On the other hand, we have above observed that every finite factor-group of G is Dedekind, in particular, it is soluble. This contradiction shows that $K = S$, i.e K is soluble. Suppose that K is non-identity. In this case $U = [K, K] \neq K$, and V/U is metabelian and non-abelian.

Being finitely generated, V/U is residually finite [26, Theorem 1]. It follows that a subgroup U is an intersection of some subgroups of finite index. On the other hand, R is an intersection of all subgroups having in V finite index. Therefore $R \leqslant Y$. This contradiction shows that $K = \langle 1 \rangle$, so that V is abelian. Since G/V is finite, G is abelian-by-finite, as required. □

Lemma 5. *Let G be a finitely generated group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that G is torsion-free and includes a normal abelian subgroup A such that G/A is a finite cyclic group. Then G is abelian.*

Proof. Let x be an element of G such that $G/A = \langle xA \rangle$. Since G is torsion-free, $1 \neq x^m \in A$ for some $m \in N$. In particular, $C = C_A(x) \neq \langle 1 \rangle$. It is not hard to prove that C is a pure subgroup of A. If $C = A$, then G is abelian, and all is proved. Therefore suppose that $C \neq A$. Clearly C is $\langle x \rangle$-invariant subgroup. The element xC has finite order in a factor-group G/C. Using Lemma 2 we obtain that $xC \in C_{G/C}(A/C)$. It follows that G/C is abelian. Since A is abelian, the choice of C shows that $C \leqslant \zeta(G)$. In particular, G is nilpotent. Being torsion-free and abelian-by-finite, G is abelian. □

Corollary 1. *Let G be a finitely generated group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that G is torsion-free and soluble-by-finite. Then G is abelian.*

Proof. By Corollary to Lemma 4 G has a normal abelian subgroup A such that G/A is finite. We have $G = \cup_{1 \leqslant j \leqslant k} A x_j$ for some elements $x_1, \ldots, x_k$. Lemma 5 shows that a subgroup $\langle A, x_j \rangle$ is abelian, $1 \leqslant j \leqslant k$. An equation $G = \cup_{1 \leqslant j \leqslant k} \langle A, x_j \rangle$ proves now that a group G has a center of finite index (see, for example, [27, Theorem 7.4]). In particular, G is an FC-group. We observe now that a torsion-free FC-group is abelian (see, for example, [27, Theorem 1.6]). □

Corollary 2. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that G is torsion-free. Then G is abelian.*

Lemma 6. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then the set of all elements of G, having finite order, is a (characteristic) subgroup of G.*

Proof. Let x, y be two arbitrary elements of finite order and let $F = \langle x, y \rangle$. Suppose that a subgroup F is infinite. By Corollary to Lemma 4 F

includes a normal abelian subgroup A of finite index. Then A is also finitely generated. In this case there exists a positive integer k such that $K = A^k$ is torsion-free and non-identity. Furthermore, A/A^k is finite, so that G/K is likewise finite. Lemma 2 shows that $[K, x] = \langle 1 \rangle = [K, y]$. An equation $F = \langle x, y \rangle$ shows that $K \leqslant \zeta(F)$. In other words, the center of G has a finite index, in particular, F is an FC-group. However, the set of all elements of finite order of an FC-group is a subgroup (see, for example, [27, Theorem 1.6]). In particular, $\langle x, y \rangle$ is finite. This contradiction proves a result. □

Corollary 1. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then every finitely generated subgroup of G is central-by-finite. In particular, G is locally FC-group.*

Proof. Let F be an arbitrary finitely generated subgroup of G. Denote by T the subset of all elements of F, having finite order. By Lemma 6 T is a characteristic subgroup of F. Corollary 1 of Lemma 5 proves that F/T is abelian. In particular, the derived subgroup $[F, F]$ is finite. It follows that F is an FC-group. However a finitely generated FC-group is central-by-finite. □

Corollary 2. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then the derived subgroup of G is periodic.*

Proof. Denote by T the subset of all elements of G, having finite order. Lemma 6 shows that T is a characteristic subgroup of G. A factor-group G/T is torsion-free and, by Corollary 2 of Lemma 5, G/T is abelian. Hence $[G, G] \leqslant T$. □

Having put together everything proved above, we receive the following main result of this work.

Theorem 1. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then the following assertions hold:*

1. *Every finitely generated subgroup of G is central-by-finite, in particular, G is a locally FC-group.*

2. *The derived subgroup of a group G is a locally finite subgroup. In particular, if G is torsion-free, then G is abelian.*

We recall that a group G is called locally graded, if every finitely generated subgroup of G has a proper subgroup of finite index. The class of locally graded groups is very broad, it includes all locally residually finite groups and all locally (soluble-by-finite) groups.

Theorem 2. *Let G be a locally graded group whose subgroups either are normal or have Chernikov derived subgroups. Then the derived subgroup of a group G is locally finite. In particular, if G is torsion-free, then G is abelian.*

Proof. Denote If a group G includes a subgroup of finite index, having Chernikov derived subgroup, then clearly G is soluble-by-finite. Therefore we can assume, that every subgroup of finite index has non-Chernikov derived subgroup. Let

$$\mathfrak{M} = \{X | X \text{ is a subgroup having non-Chernikov derived subgroup}\},$$
$$R = \cap\mathfrak{M}.$$

Thus, if $H \in \mathfrak{M}$, then H is normal in G and G/H is a Dedekind group by Lemma 1. By a classical Remak theorem G/R is isomorphic to some subgroup of $\prod_{X\in\mathfrak{M}} G/X$. Since every Dedekind group either is abelian or nilpotent of class 2, either G/R is abelian or nilpotent of class 2. (We can noted, that it is possible that $G = R$). If H is a proper subgroup of R, then by a construction of $R, [H, H]$ is a Chernikov subgroup. Let H be an arbitrary finitely generated subgroup of R. Since G is locally graded, H includes a proper normal subgroup L of finite index. In particular, L is a proper subgroup of R and by above L has a Chernikov derived subgroup. It shows that L is soluble-by-finite. Corollary to Lemma 4 shows that L is abelian-by finite. Hence and H is abelian-by-finite. If $H = R$, then we obtain that R is abelian-by-finite, so that G is soluble-by-finite. Suppose now that R has no finite set of generators. By above proved R is locally (abelian-by-finite). By Theorem 1 a subgroup $K = [R, R]$ is locally finite. Then G/K is soluble. Using again Theorem 1 we obtain that a subgroup $P/K = [G/K, G/K]$ is locally finite. Since K and P/K is locally finite, P is locally finite. Since G/P is abelian, $[G, G] \leqslant P$, in particular, $[G, G]$ is locally finite. □

Corollary. *Let G be a locally graded group whose subgroups either are normal or have Chernikov derived subgroups. If G is periodic, then G is locally finite.*

References

[1] Dedekind R. Über Gruppen, deren sammtliche Teiler Normalteiler sind // Math. Annalen.– 1897.– 48.– S. 548–561.

[2] Baer R. Situation der Untergruppen und Struktur der Gruppe //S.– B. Heidelberg Akad.– 1933.– 2.– S. 12–17.

[3] Schmidt O.Y. Groups which have one class of not invariant subgroups // Mat. sbor.– 1926.– 33.– p. 161–172.

[4] Schmidt O.Y. Groups with two classes of not invariant subgroups // Proc. of sem. on the theory of groups – M.– L.: NAUKA.– 1938.– p. 7–26.

[5] Chernikov N.S. (1967) Groups with Prescribed Properties of Systems of Finite Subgroups // Ukrain. Mat. Zh.– 1967.– 19(6)– p. 111–131.

[6] Chernikov N.S. Groups with Prescribed Properties of Systems of Subgroups // NAUKA: Moskow– 1980.

[7] Artemovich, O.D., Kurdachenko, L.A.: Groups which are rich on ?– subgroups // Proc. Lvov Univ. Ser. Mech. y Mathematic.– 2003.– 67.– p. 218–237

[8] Romalis G.M., Sesekin N.F. About meta– Hamiltonian groups I // Mat. Not. Ural. Univ.– 1966.– 5(3).– p. 45–49.

[9] Sesekin N.F., Romalis G.M. About meta– Hamiltonian groups II // Mat. Not. Ural. Univ.– 1968.– 6(5).– p. 50–53.

[10] Romalis G.M., Sesekin N.F. About meta– Hamiltonian groups III // Mat. Not. Ural. Univ.– 1970.– 7(3).– p. 195–199.

[11] Nagrebeckiy V.T. Finite nonnilpotent groups with any nonabelean subgroups which invariant // Mat. Not. Ural. Univ.– 1967.– 6(1).– p. 80–88.

[12] Mahnev A.A. About finite meta– Hamiltonian groups // Mat. Not. Ural. Univ.– 1976.– 10(1).– p. 60–75.

[13] Kuzennyi, N.F. and Semko N.N. The structure of solvable nonnilpotent meta– Hamiltonian groups // Mat. Zametki– 1983.– 34(2).– p. 179–267.

[14] Kuzennyi, N.F. and Semko N.N. The structure of solvable meta– Hamiltonian groups // Dokl. Akad. Nauk Ukrain. SSR Ser. A.– 1985.– 2(6– 8).– p. 87.

[15] Semko N.N., Kuzennyi N.F. The structure of infinite nilpotent periodic meta– Hamiltonian groups. In The Structure of Groups and Subgroups Characterizations, pp. 101– 11, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev.– 1984

[16] Kuzennyi, N.F. and Semko N.N. The structure of periodic metabelian and meta– Hamiltonian groups with a non– elementary commutator group // Ukrain. Mat. Zh.– 1987.– 39(2)– p. 180–5, 271.

[17] Semko N.N., Kuzennyi N.F. (1989). The structure of metacyclic meta– Hamiltonian groups // In current Analysis and its Applications, pp. 173–83, 226. "Naukova Dumka", Kiev.

[18] Kuzennyi, N.F. and Semko N.N. The structure of periodic nonabelian meta– Hamiltonian groups with an elementary commutator subgroup of rank three // Ukrainian Math. J.– 1989.– 41(2).– p. 153–161.

[19] Kuzennyi, N.F. and Semko N.N. Meta– Hamiltonian groups with elementary commutator subgroup of rank two // Ukrainian Math. J.– 1990.– 42(2).– p. 149–54.

[20] Kurdachenko L., Otal J., Russo A., Vincenci G. The local structure of groups whose non– normal subgroups have finite conjugacy classes //Advanced in Group Theory 2002. Proceedings of the Intensive Bimester Dedicated to the Memory of Reinhold Baer, Napoly, May– June 2002.– Roma: Aracne, 2003.– p. 93–110.

[21] Kurdachenko L., Otal J., Russo A., Vincenci G. Groups whose non– normal subgroups have finite conjugate classes // Mathematical Proceedings of the Royal Irish Academy.– December 2004.– 104A, N. 2,– p.177–189.

[22] Robinson D.J.S. Finiteness conditions and generalized soluble groups: Part 1.– Berlin: Springer, 1972.– 210 p.

[23] Segal D. Polycyclic groups.– Cambridge: Cambridge Univ. Press, 1983.– 289 p.

[24] Hall P. Finiteness conditions for soluble groups // Proc. London Math. Soc.– 1954.– 4.– p. 419–436.

[25] Robinson D.J.S. A course in the theory of groups.– New York: Springer, 1982.– 481 p.

[26] Hall P. On the finiteness of certain soluble groups // Proc. London Math. Soc.– 1959.– 9.– p. 595–632.

[27] Tomkinson M.I. *FC*-groups.– Boston: Pitman, 1984.– 171 p.

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Received by the editors: 25.05.2009
and in final form 25.06.2009.